Dynamic Indexability
and
Lower Bounds for Dynamic One-Dimensional Range Query Indexes

Ke Yi

HKUST

First Annual SIGMOD Programming Contest (to be held at SIGMOD 2009)

- "Student teams from degree granting institutions are invited to compete in a programming contest to develop an indexing system for main memory data."

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- We think these problems are so basic that every DB grad student should know, but do we really have the answer?

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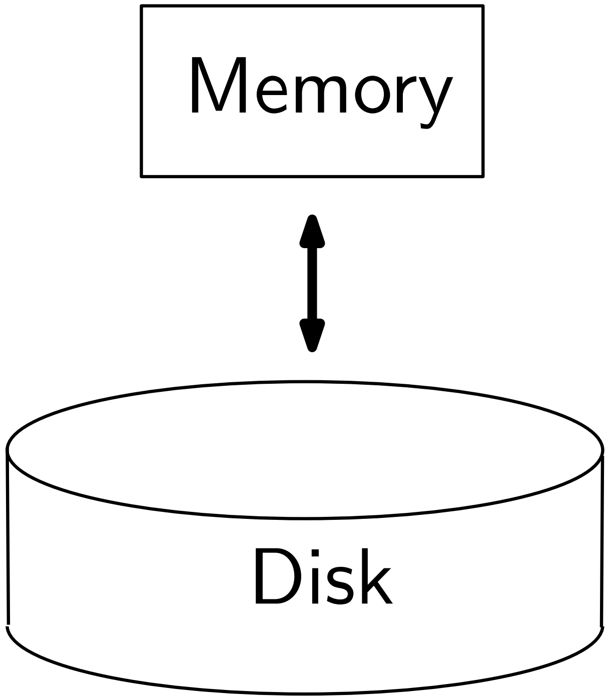
External memory model (I/O model):

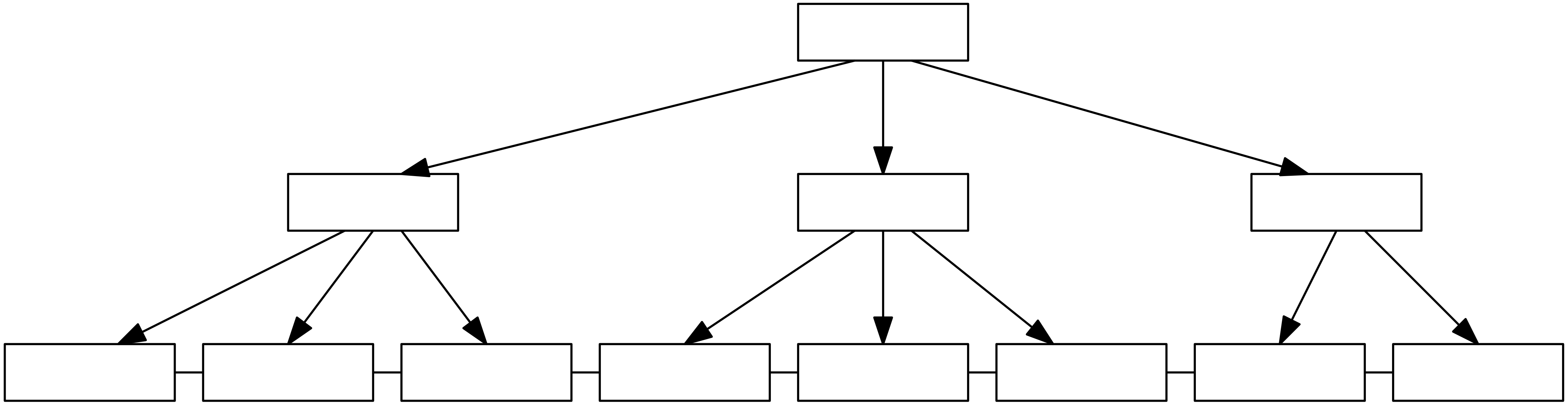


Memory of size m

Each I/O reads/writes a block

Disk partitioned into blocks of size b

The B-tree



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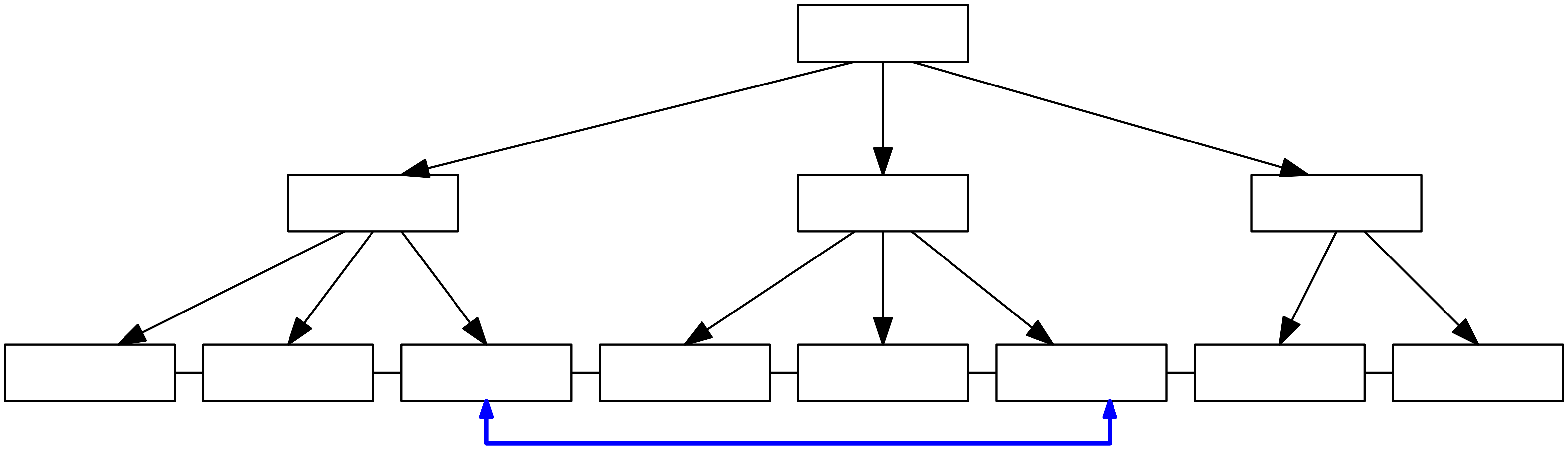
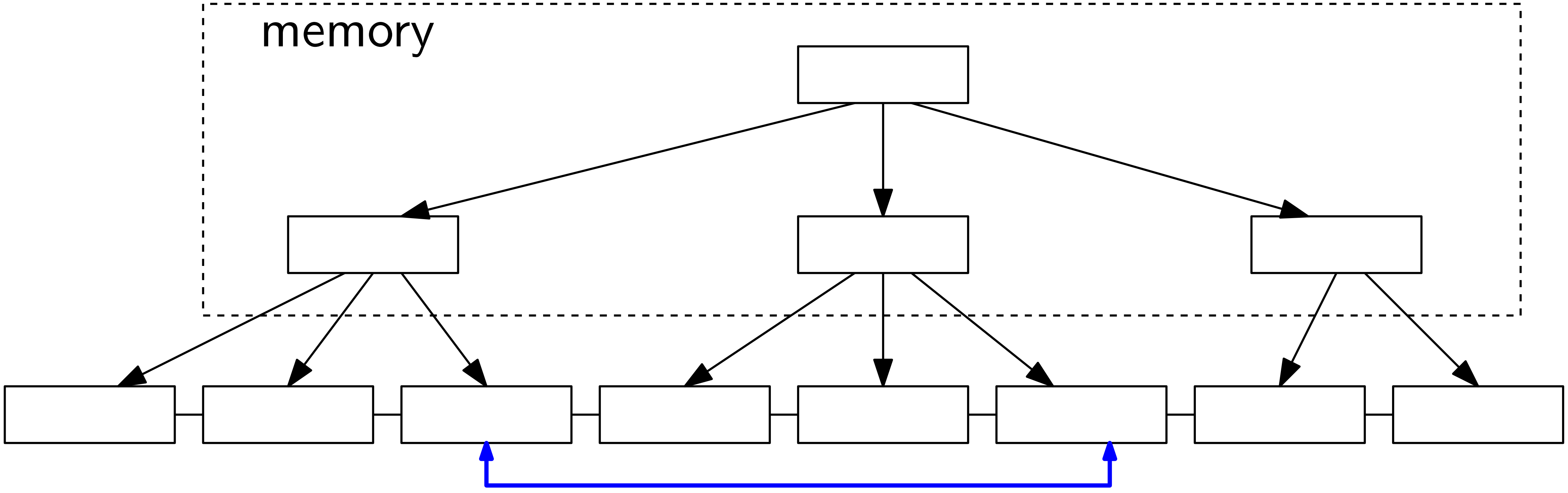


A range query in $O(\log_b n + k/b)$ I/Os

k: output size

The B-tree



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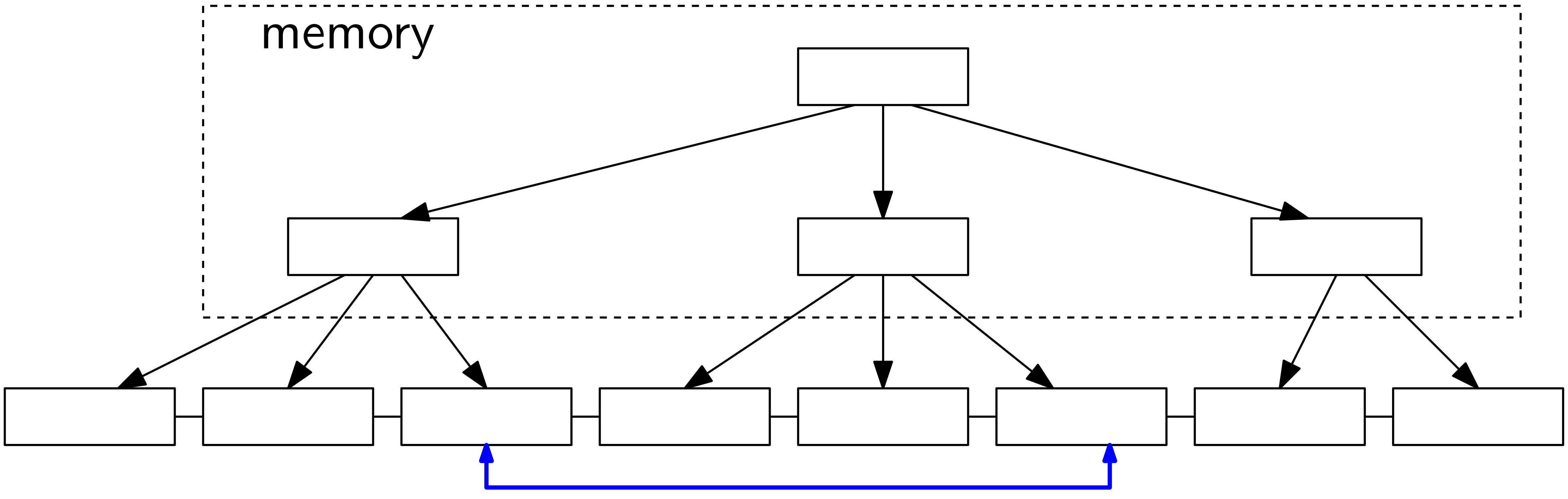

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The height of B-tree never goes beyond 5 (e.g., if $b = 100$, then a B-tree with 5 levels stores n = 10 billion records). We will assume $\log_b \frac{n}{m} = O(1)$.

Now Let's Go Dynamic

- Focus on insertions first: Both the B-tree and hash table do a search first, then insert into the appropriate block
  - B-tree: Split blocks when necessary
  - Hashing: Rebuild the hash table when too full; *extensible hashing* [Fagin, Nievergelt, Pippenger, Strong, 79]; *linear hashing* [Litwin, 80]

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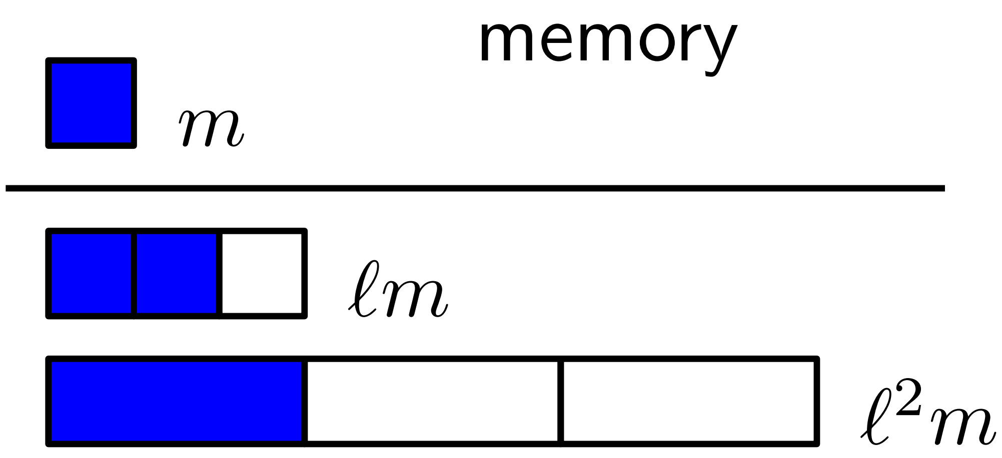
- Cannot hope for lower than 1 I/O per insertion only if the changes must be committed to disk right away (necessary?)
  - Otherwise we probably can lower the amortized insertion cost by buffering, like numerous problems in external memory, e.g. stack, priority queue,... All of them support an insertion in $O(1/b)$ I/Os — the best possible

Dynamic B-trees for Fast Insertions

- *LSM-tree* [O'Neil, Cheng, Gawlick, O'Neil, 96]: Logarithmic method + B-tree



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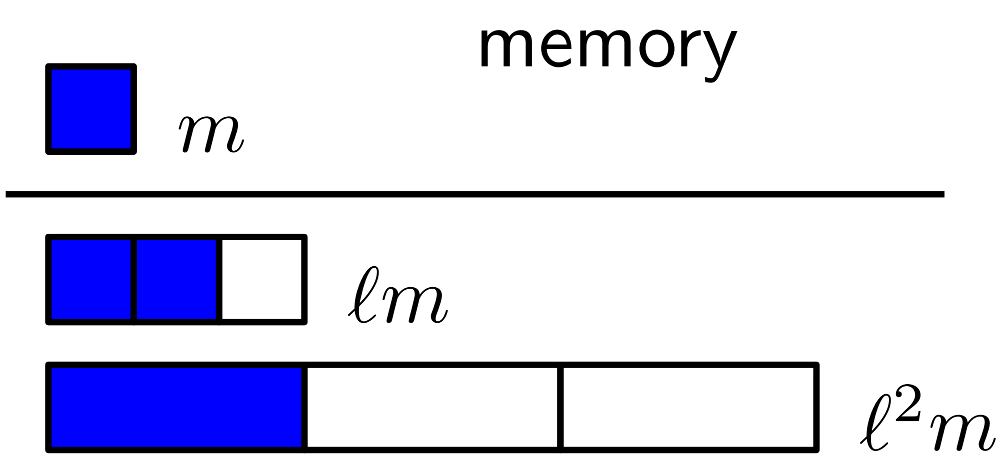

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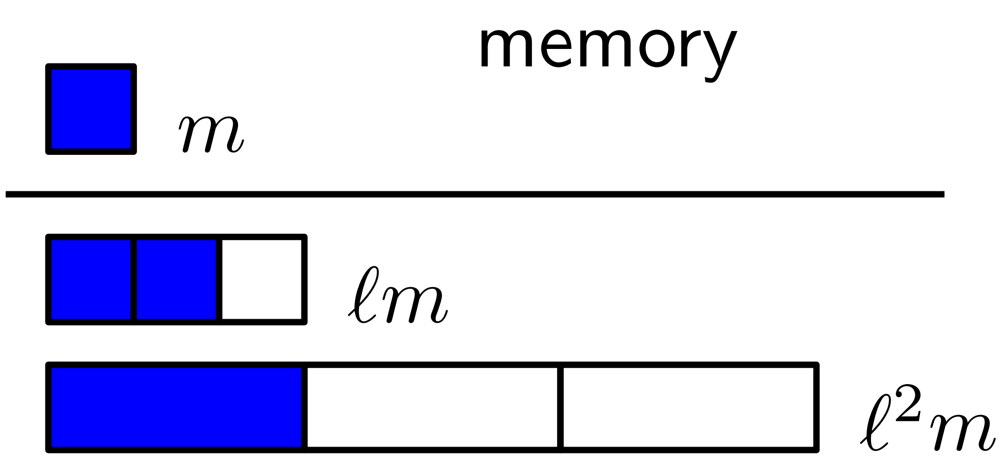

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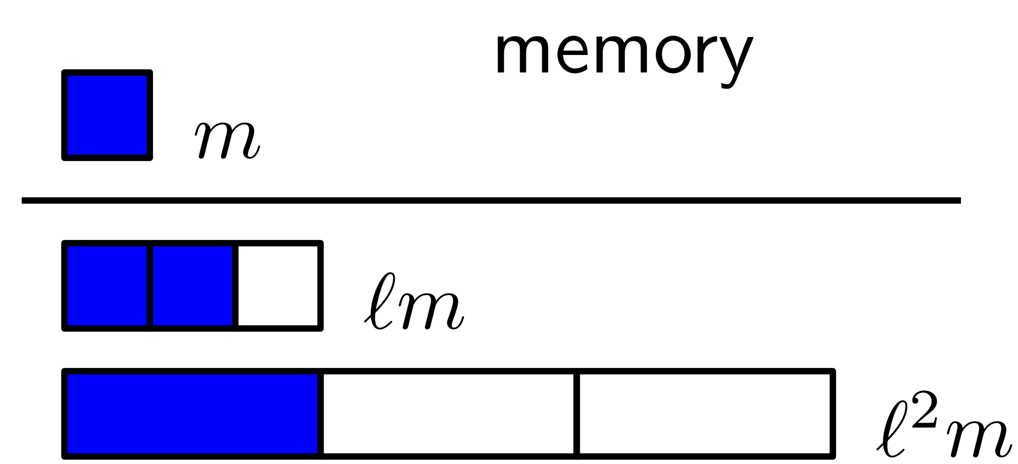

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  - Query: $O(\ell\log_\ell \frac{n}{m} + \frac{k}{b})$
- Usually ℓ is set to be a constant, then they both have $O(\frac{1}{b}\log \frac{n}{m})$ insertion and $O(\log \frac{n}{m} + \frac{k}{b})$ query

More Dynamic B-trees

- *Buffer tree* [Arge, 95]
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- No further development in the last 10 years. So, seems we can't do better, can we?

Main Result

For any dynamic range query index with a query cost of $q+O(k/b)$ and an amortized insertion cost of u/b, the following tradeoff holds

$$\begin{cases} q \cdot \log(u/q) = \Omega(\log b), & \text{for } q < \alpha \ln b, \alpha \text{ is any constant;} \\ u \cdot \log q = \Omega(\log b), & \text{for all } q. \end{cases}$$

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The technique of [Brodal, Fagerberg, 03] for the predecessor problem can be used to derive a tradeoff of

$$q \cdot \log(u \log^2 \frac{n}{m}) = \Omega(\log \frac{n}{m}).$$

Lower Bound Model: Dynamic Indexability

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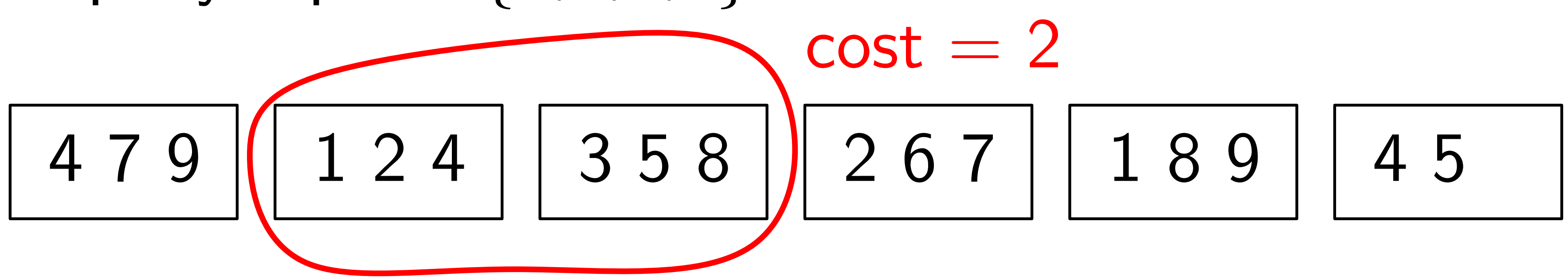

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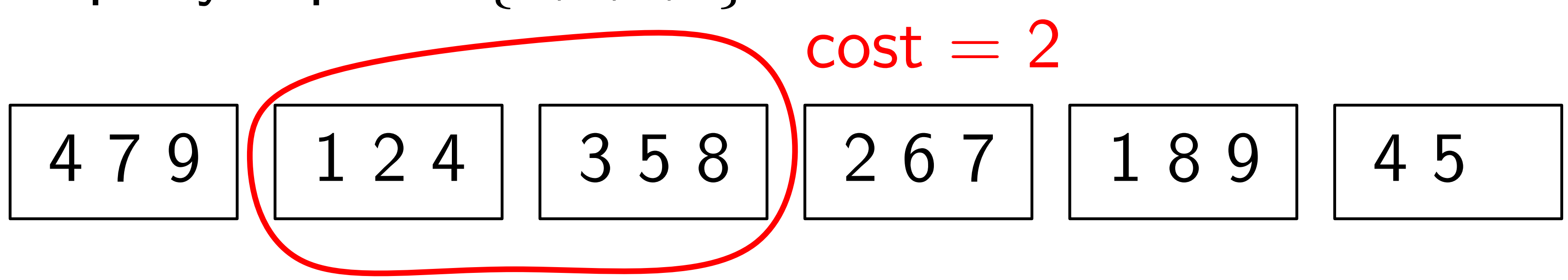

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- Similar in spirit to popular lower bound models: cell probe model, semigroup model

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  - Adding dynamization makes it much more interesting!

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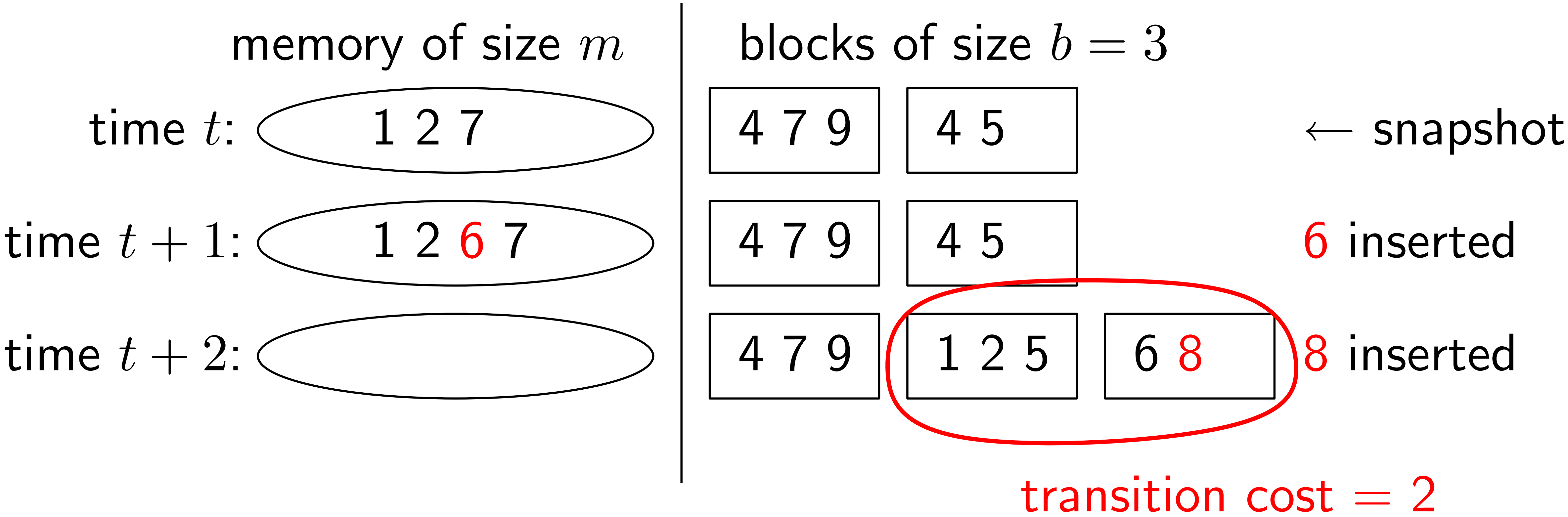

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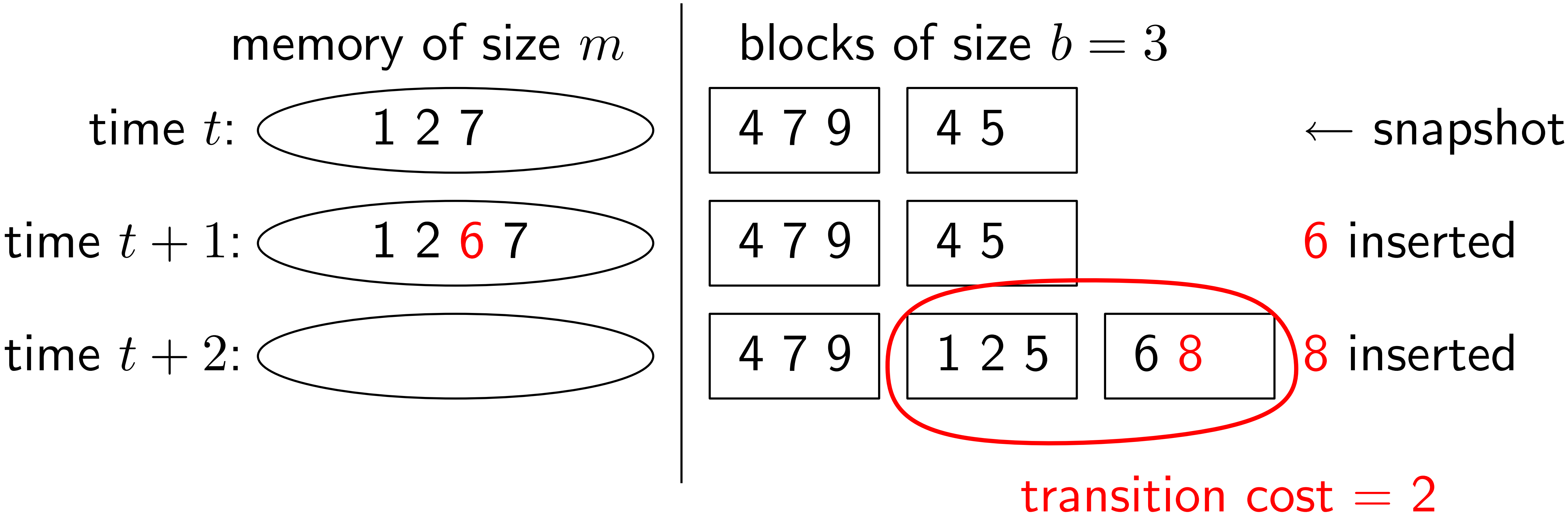

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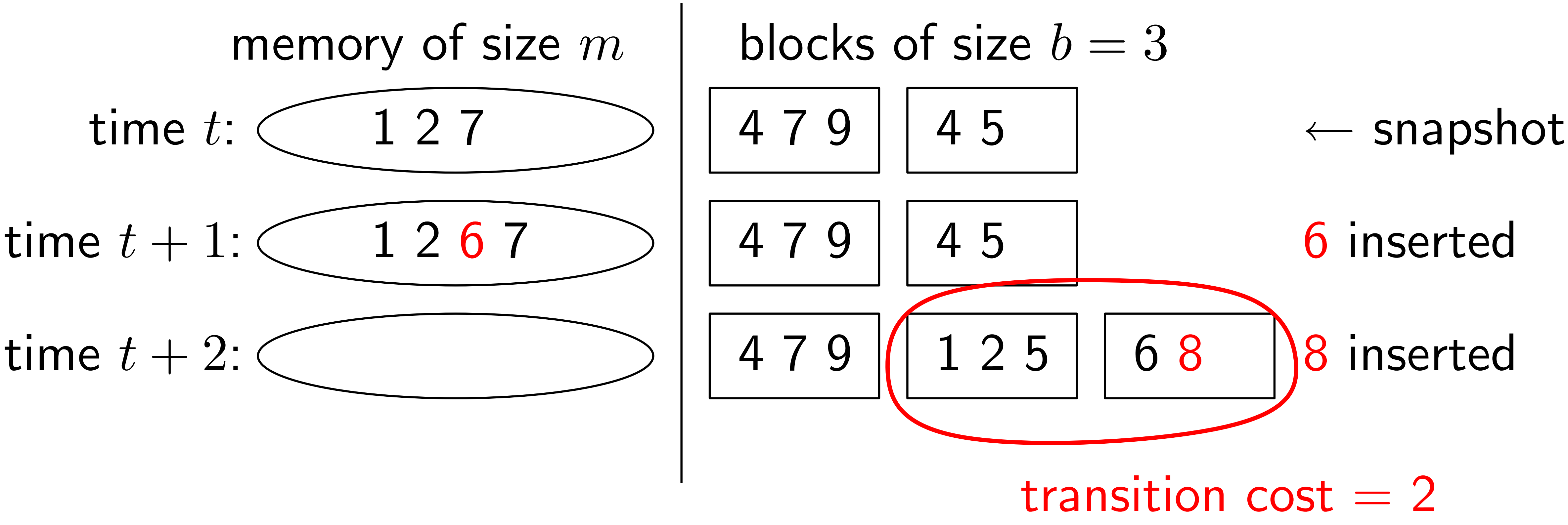

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- Update cost: $u =$ the average transition cost per b insertions

Main Result Obtained in Dynamic Indexability

THEOREM: For any dynamic 1D range query index with access overhead A and update cost u, the following tradeoff holds, provided $n \geq 2mb^2$:

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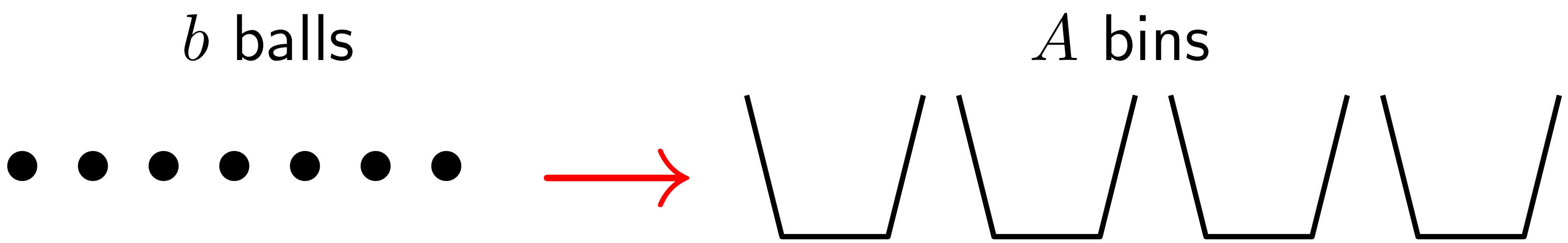
The lower bound doesn't depend on the redundancy r!

The Ball-Shuffling Problem



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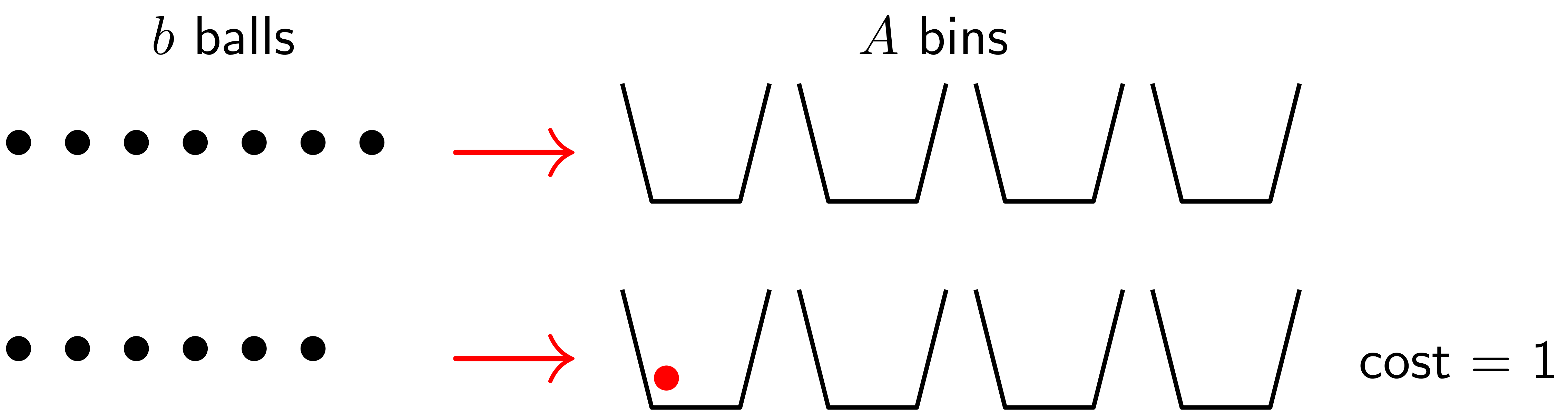

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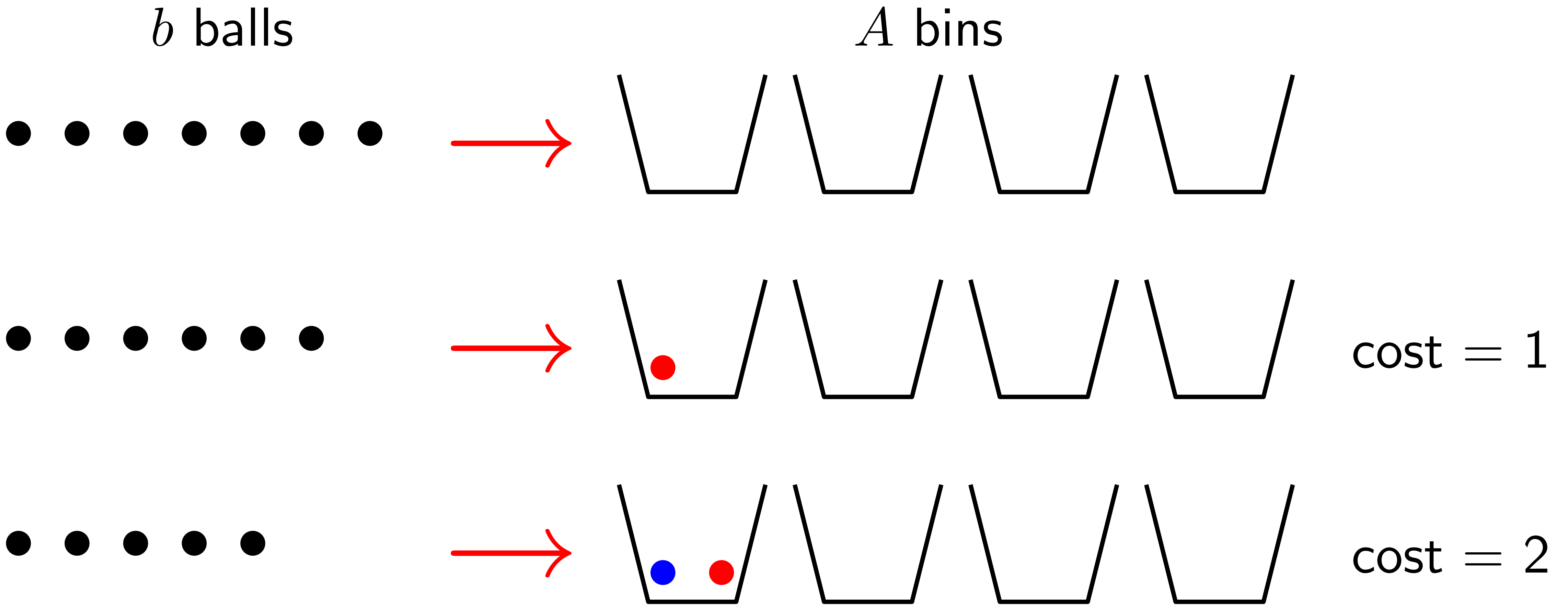
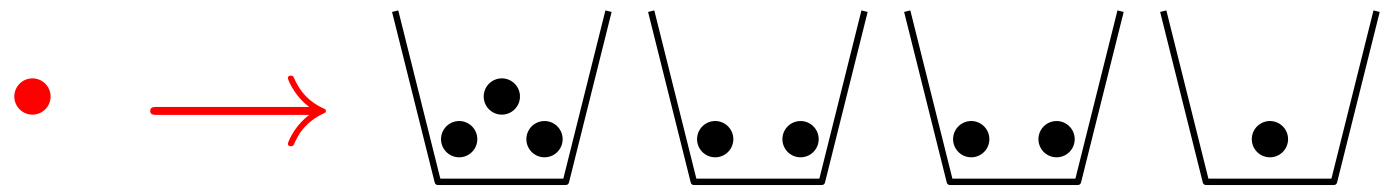


cost of putting the ball directly into a bin $=$ # balls in the bin $+ 1$

The Ball-Shuffling Problem

b balls

A bins



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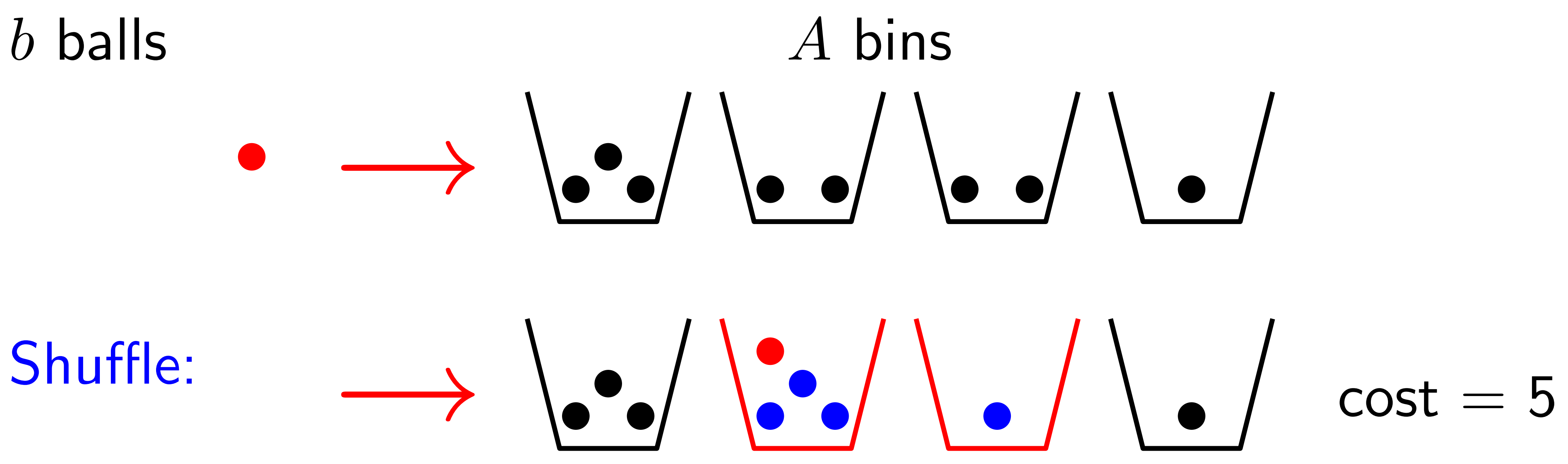

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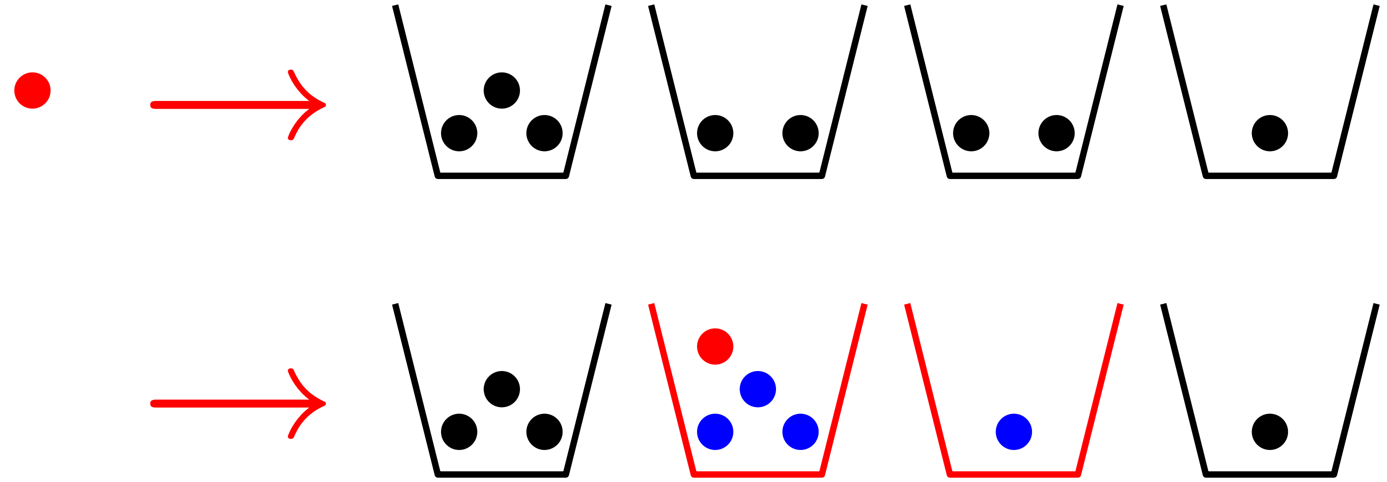

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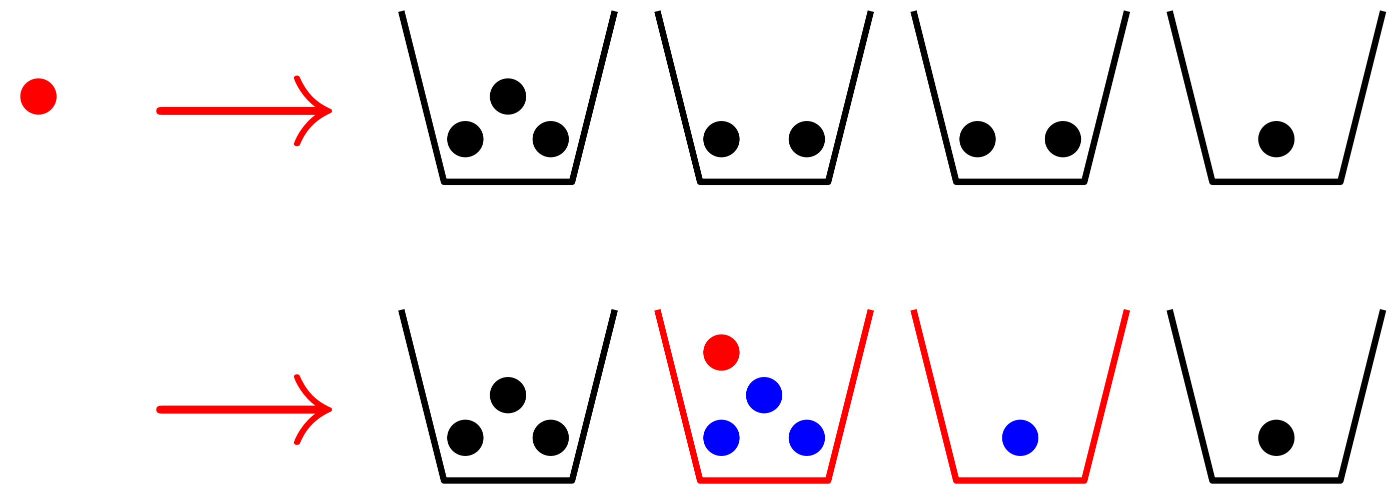

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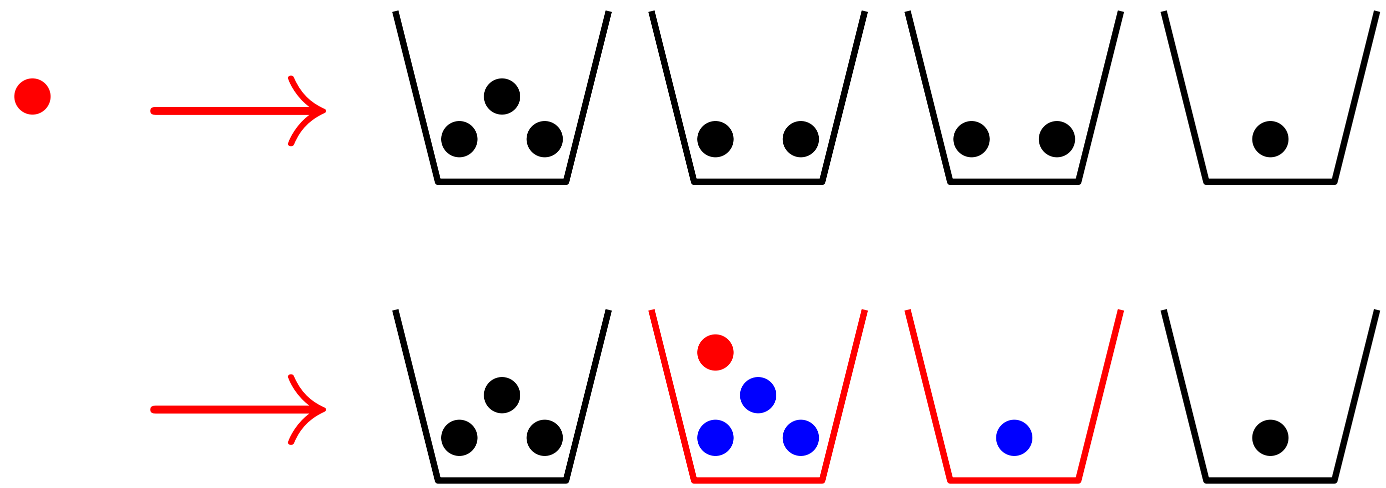

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Goal: Accommodating all b balls using A bins with minimum cost

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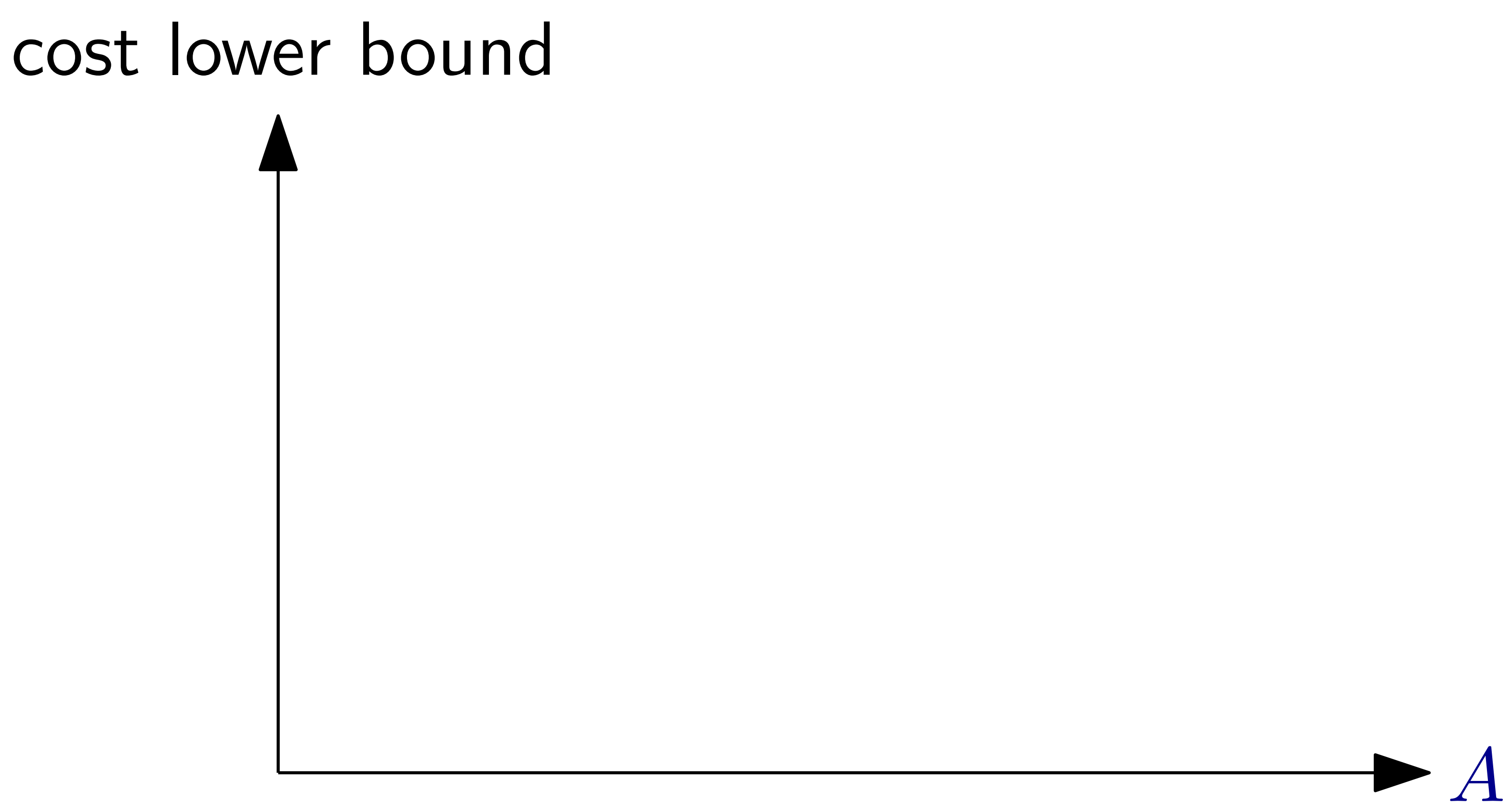

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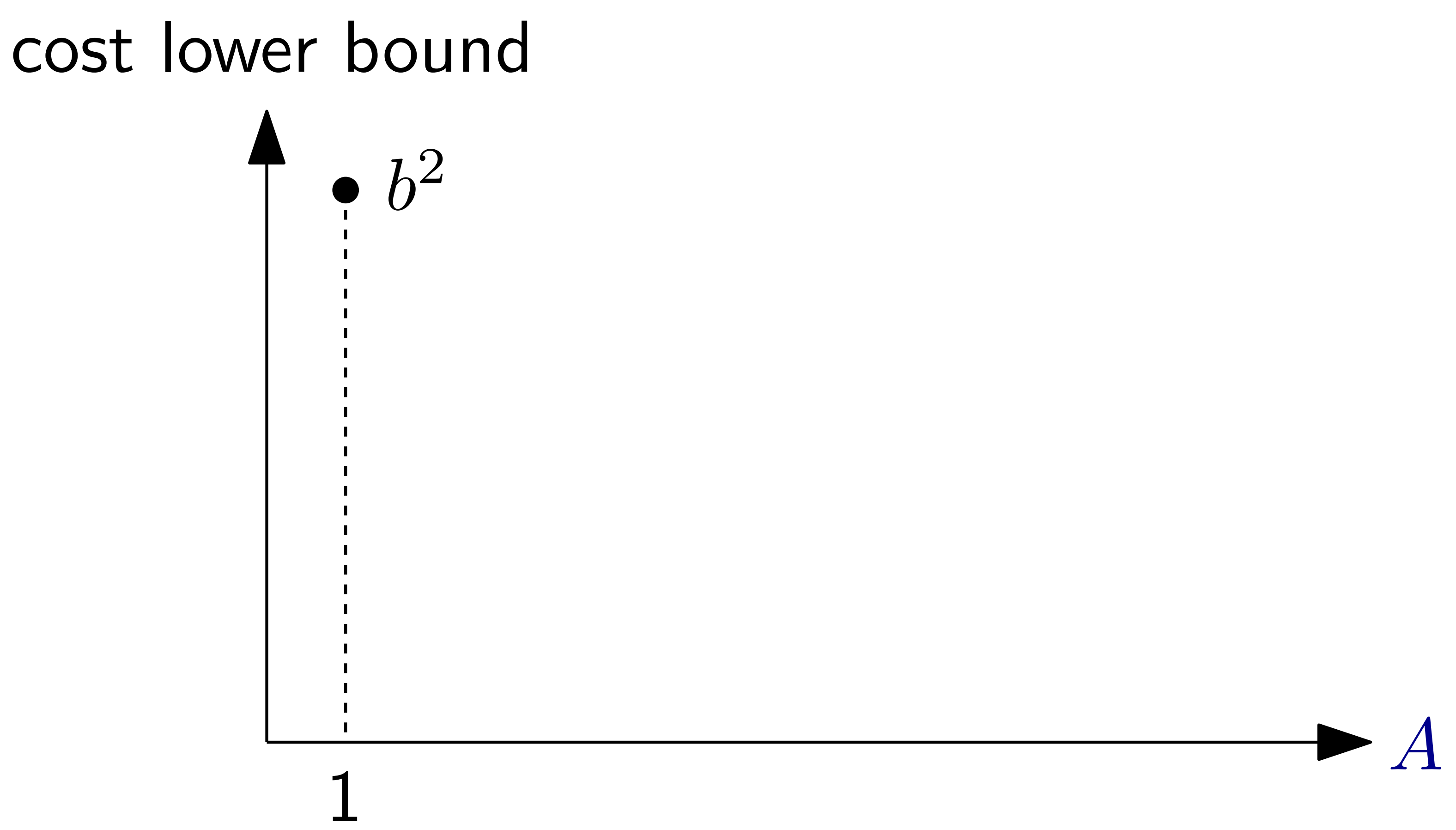

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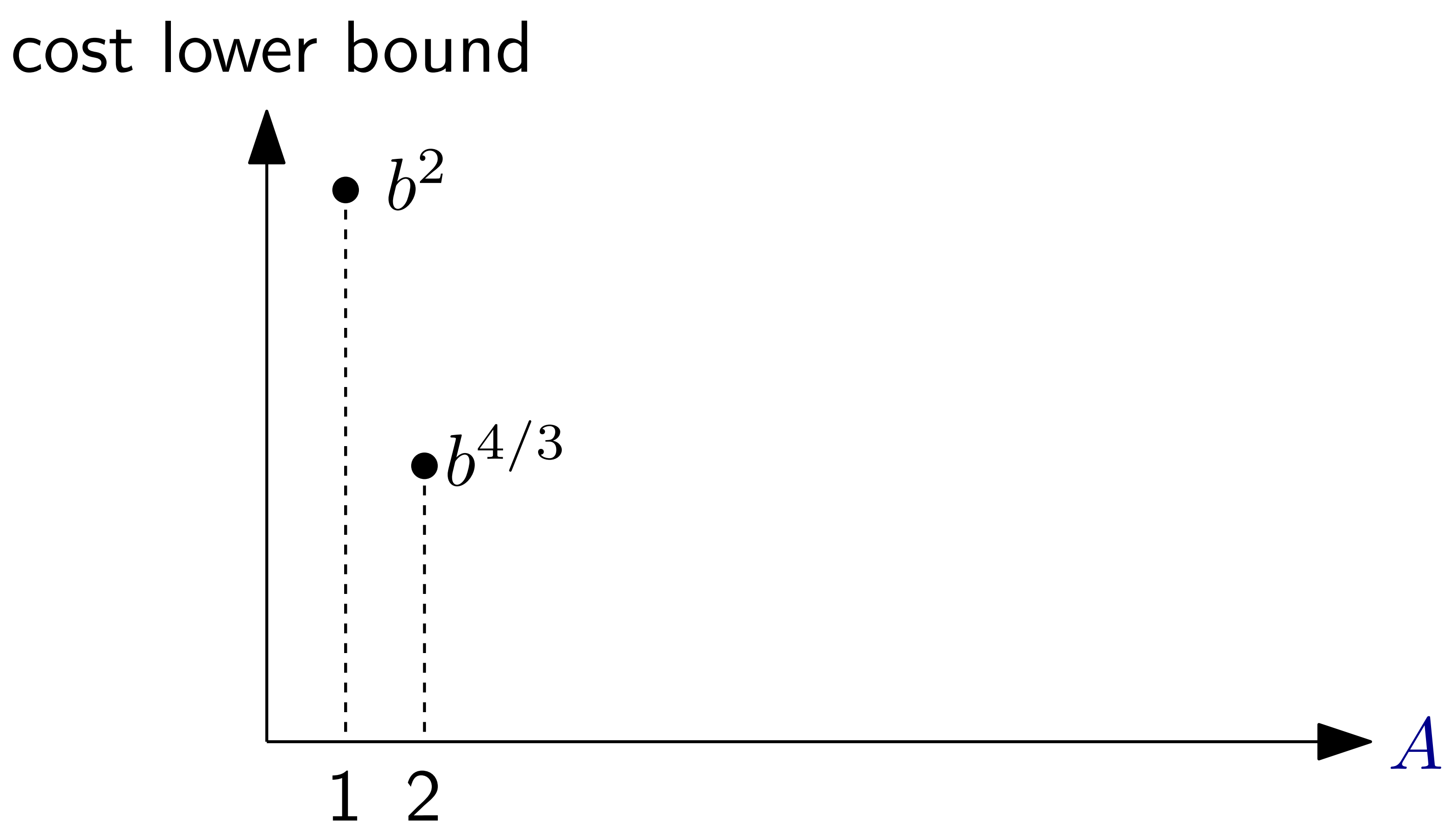

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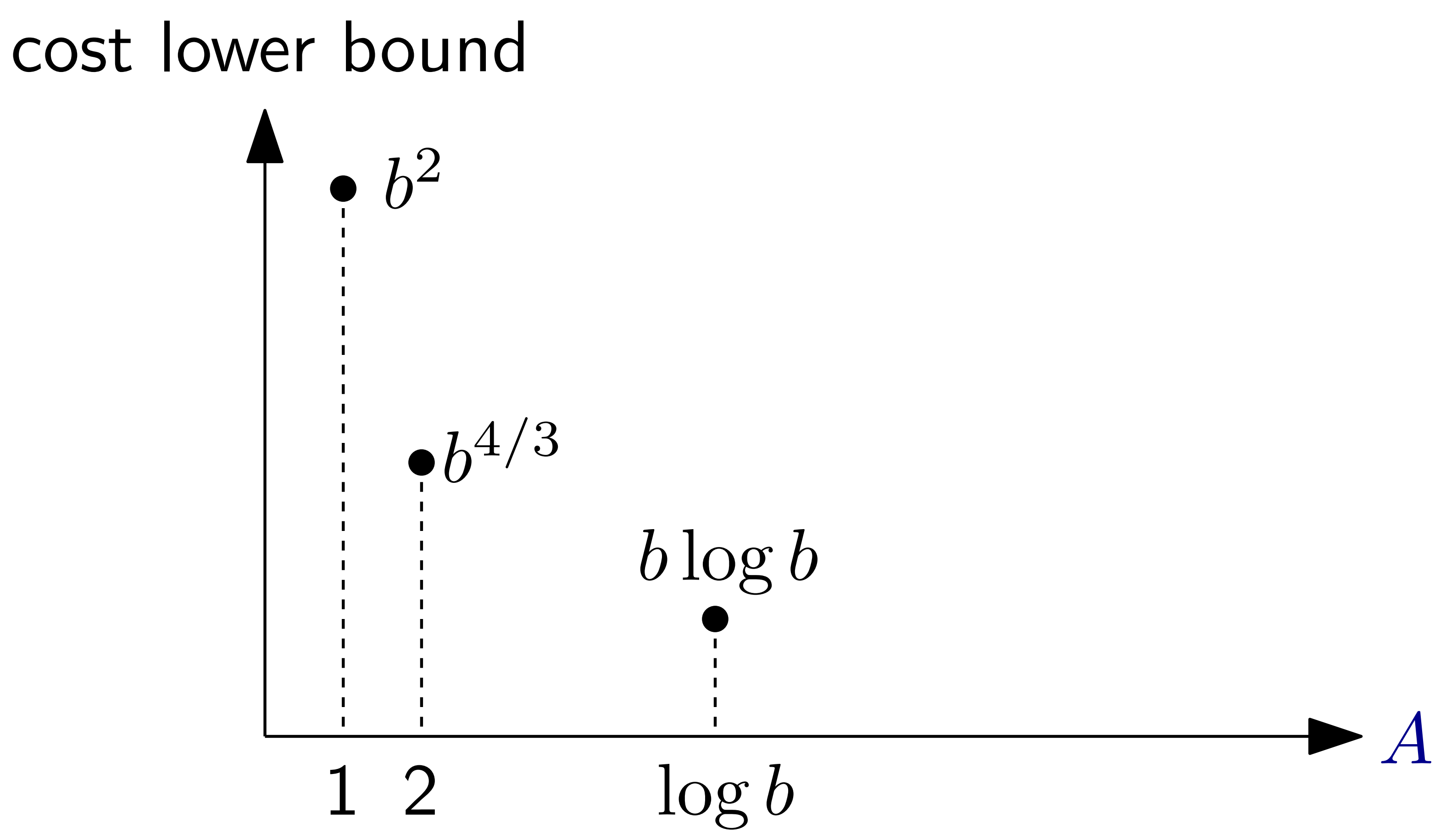

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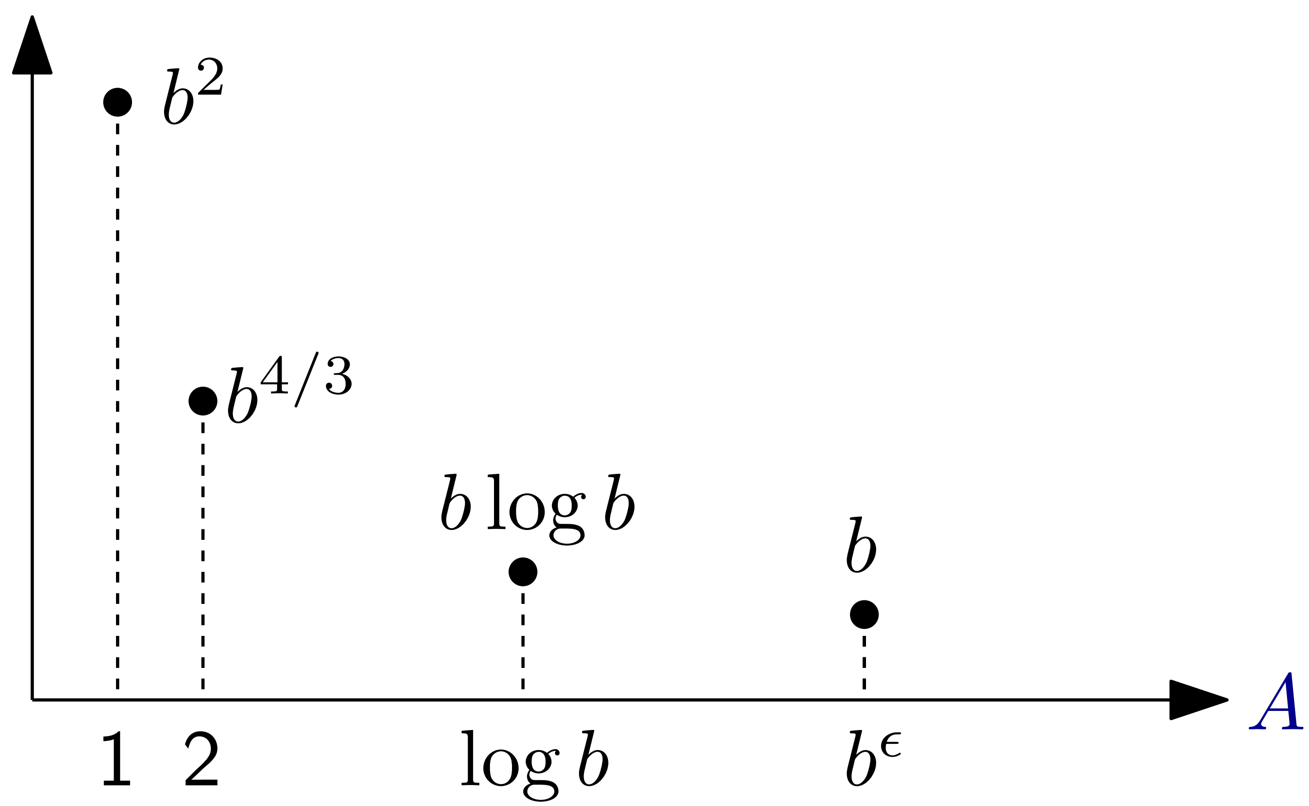

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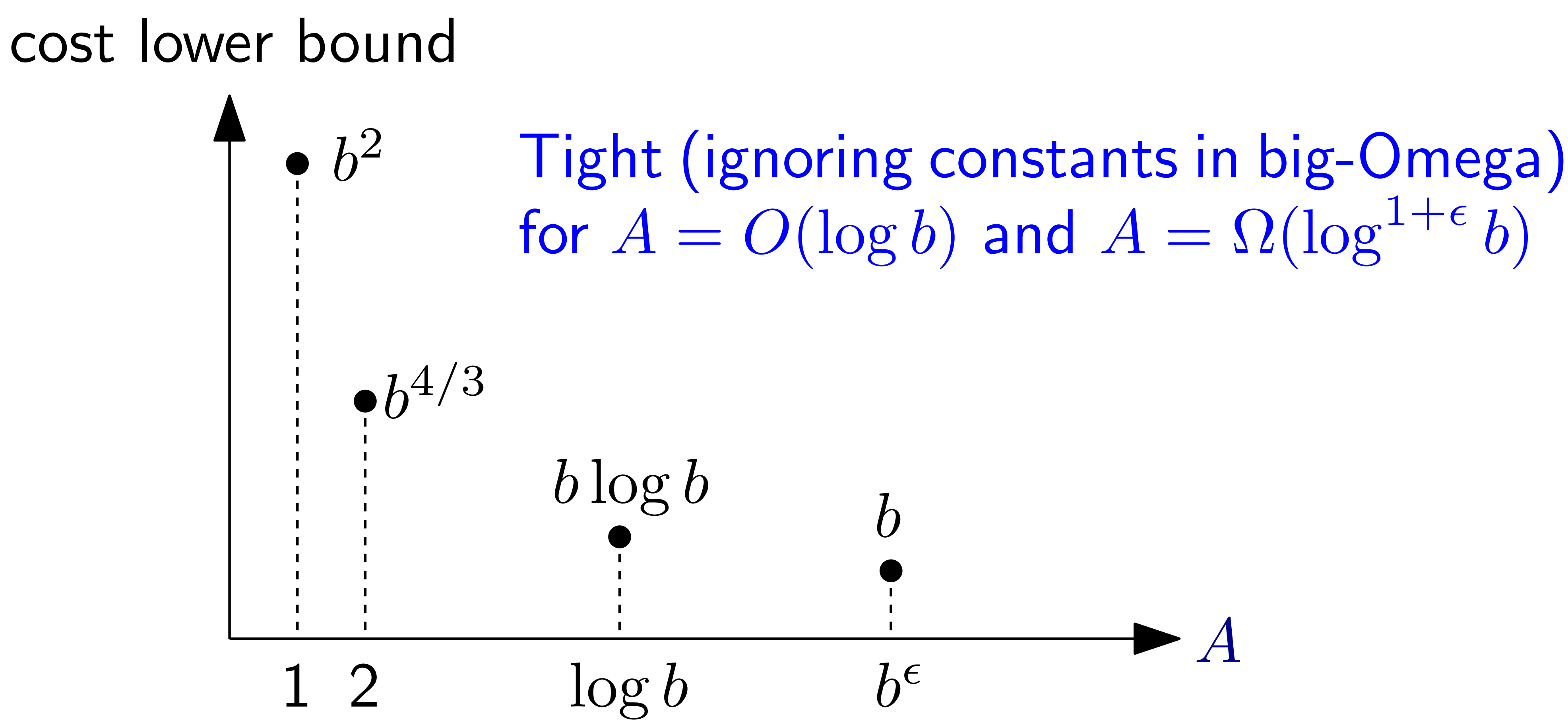
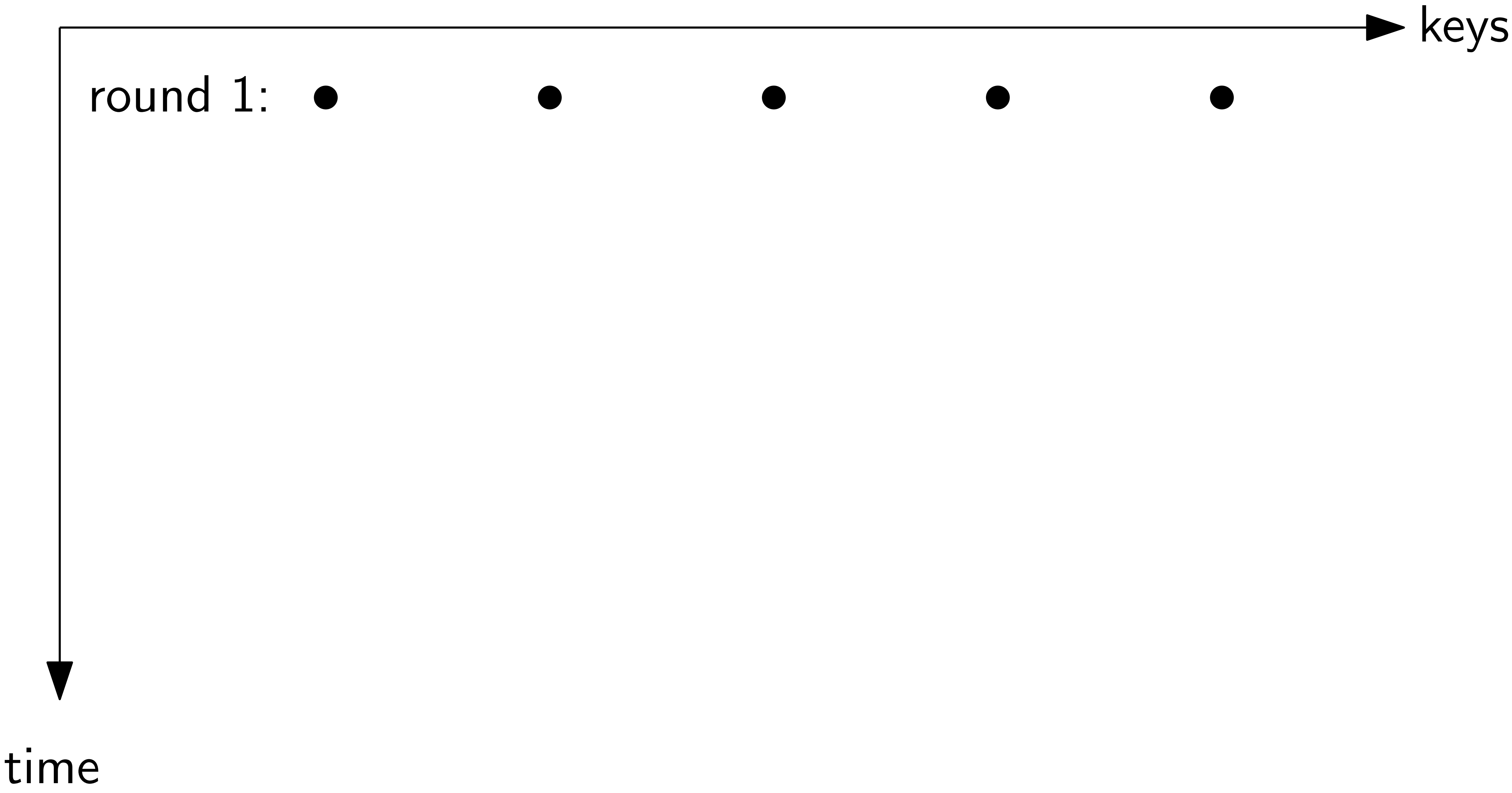


The Workload Construction



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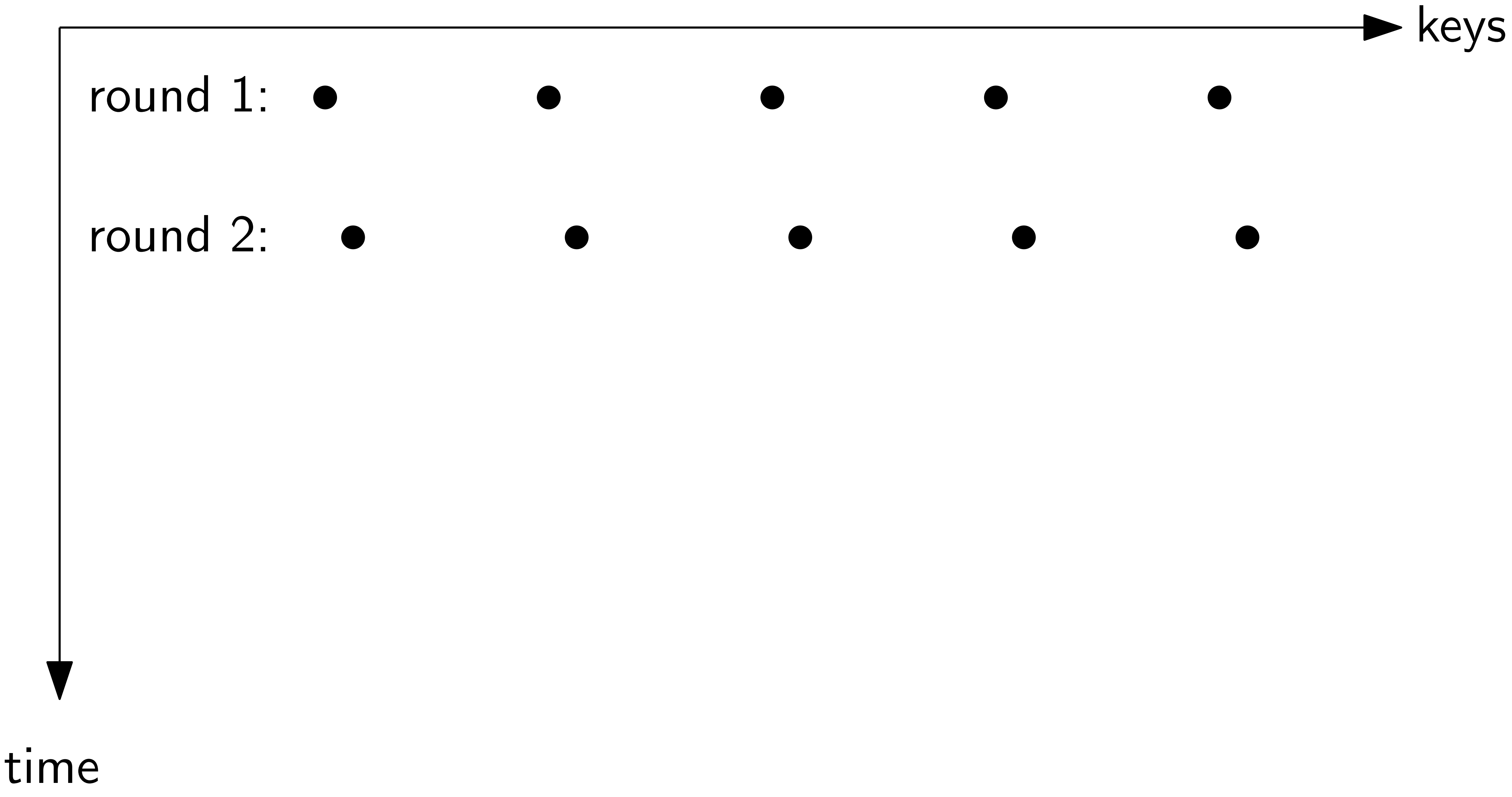

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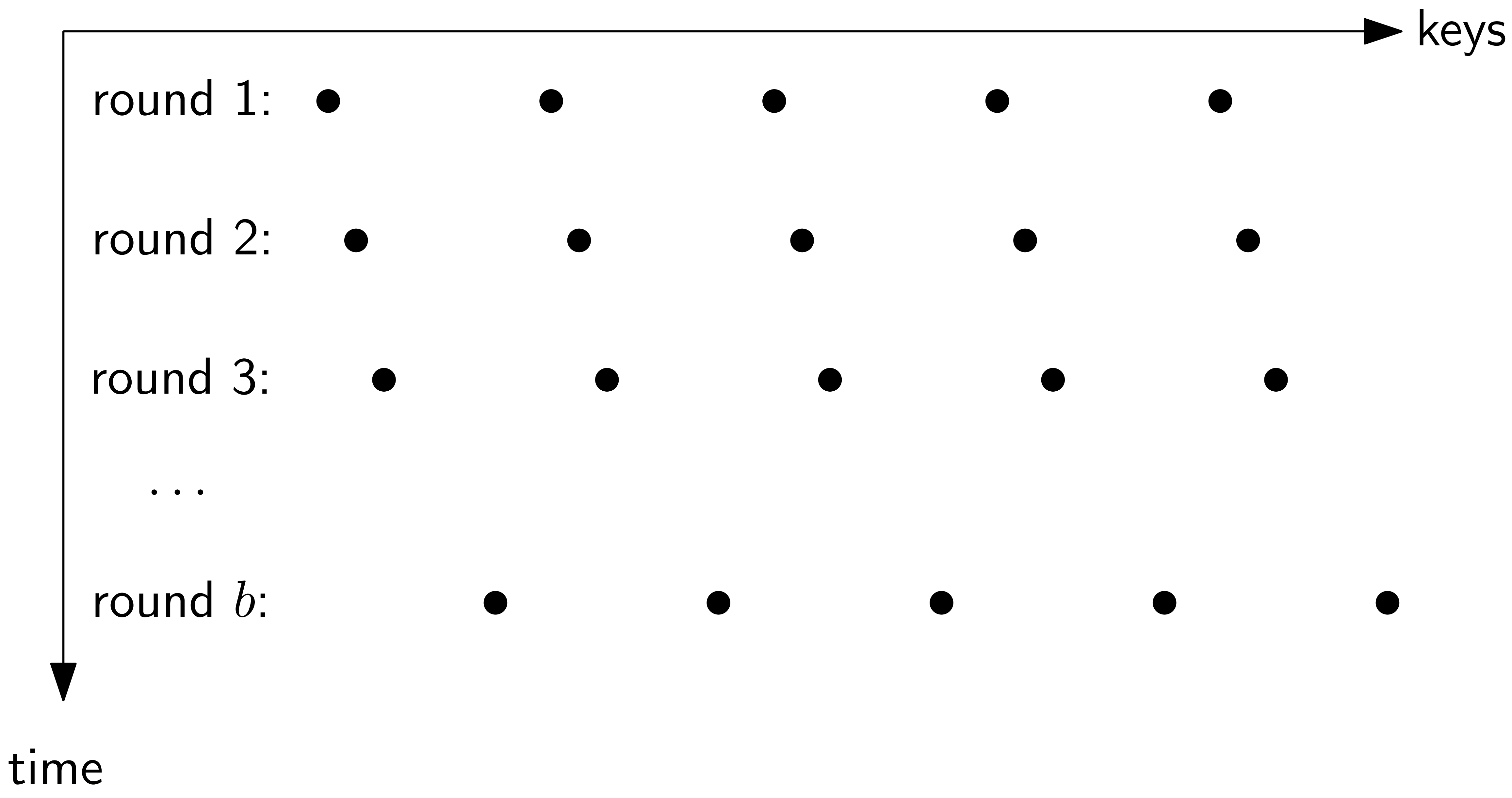

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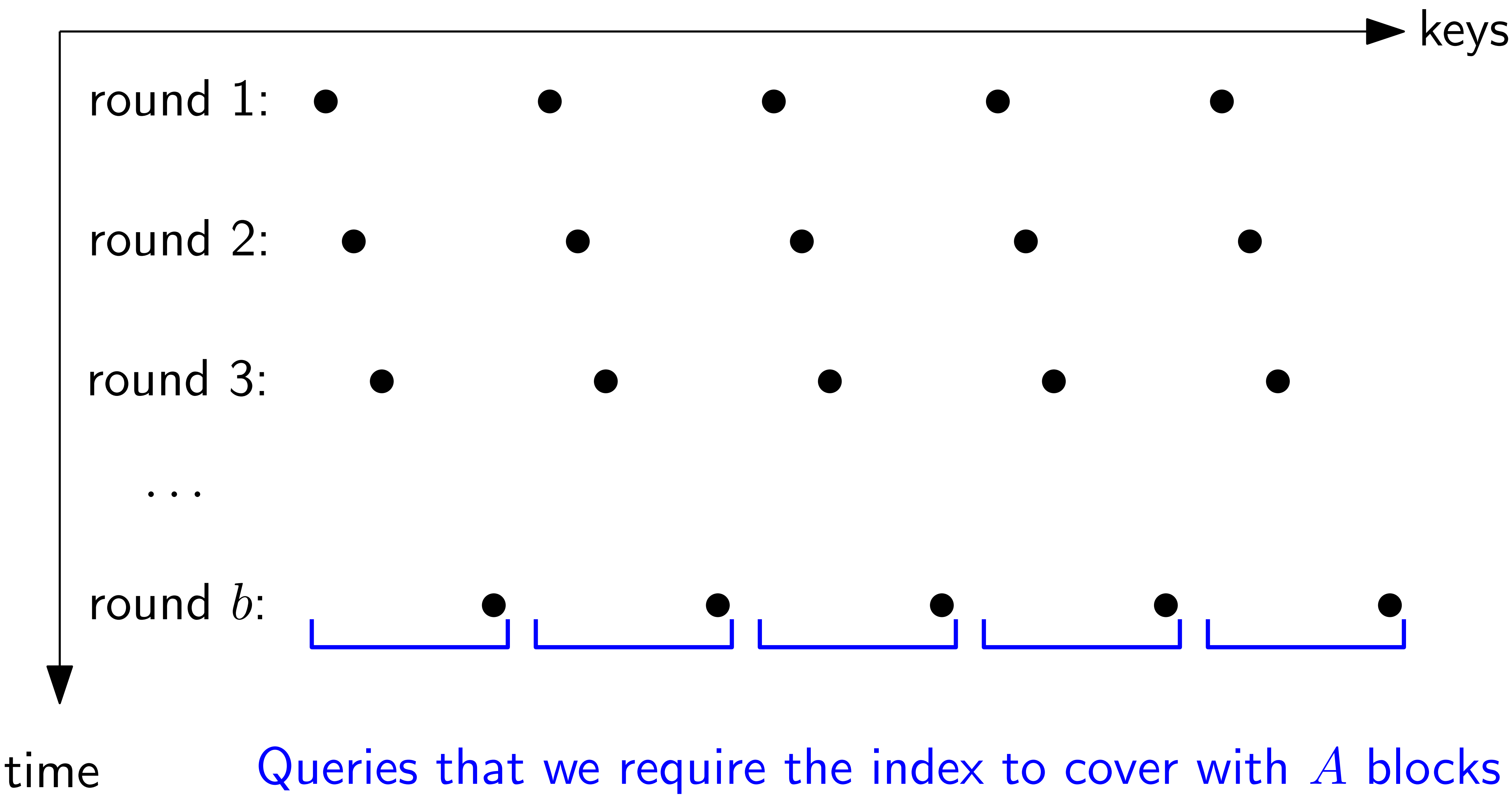

Queries that we require the index to cover with A blocks

$\#$ queries $\geq 2mb$

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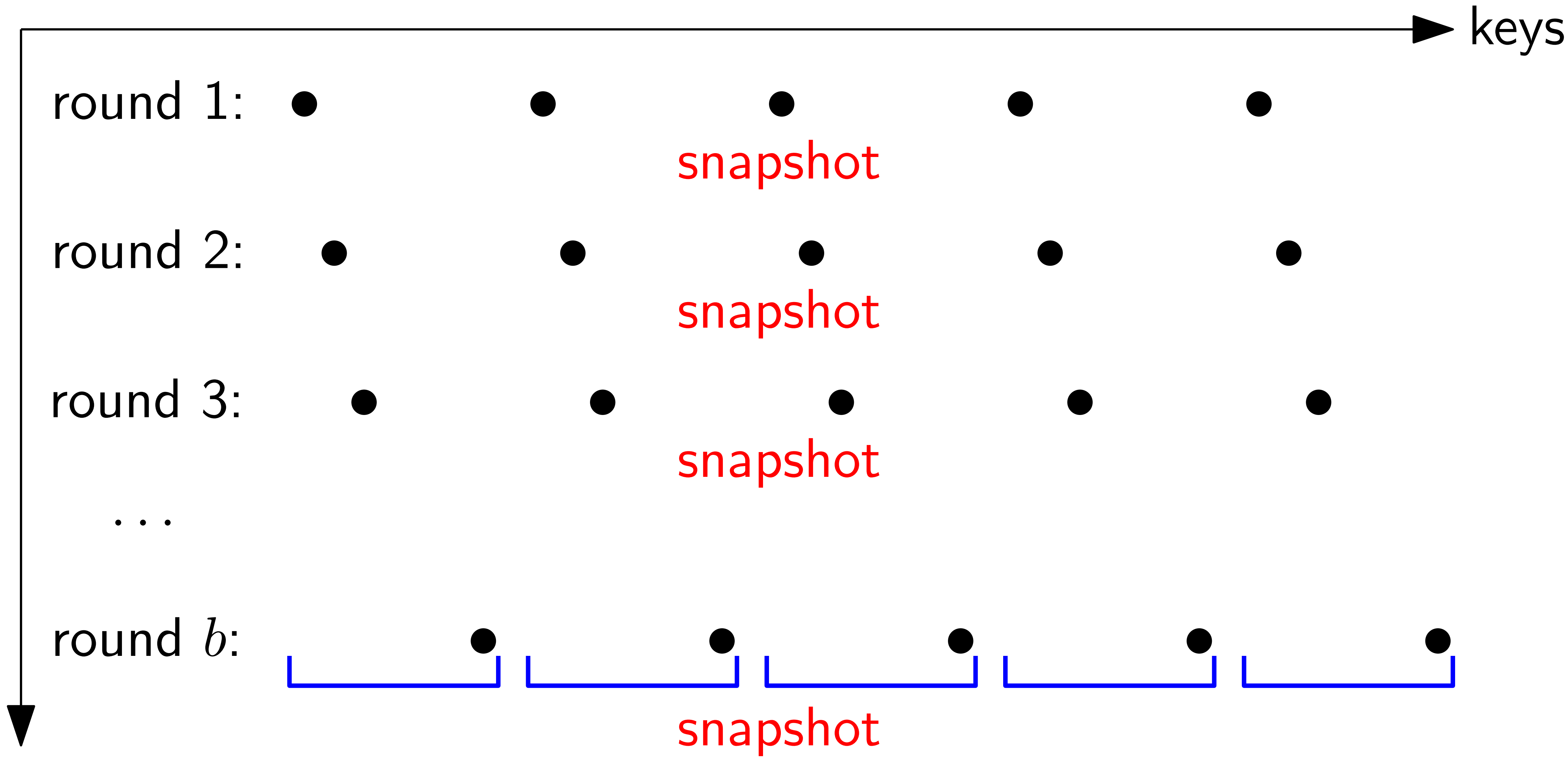

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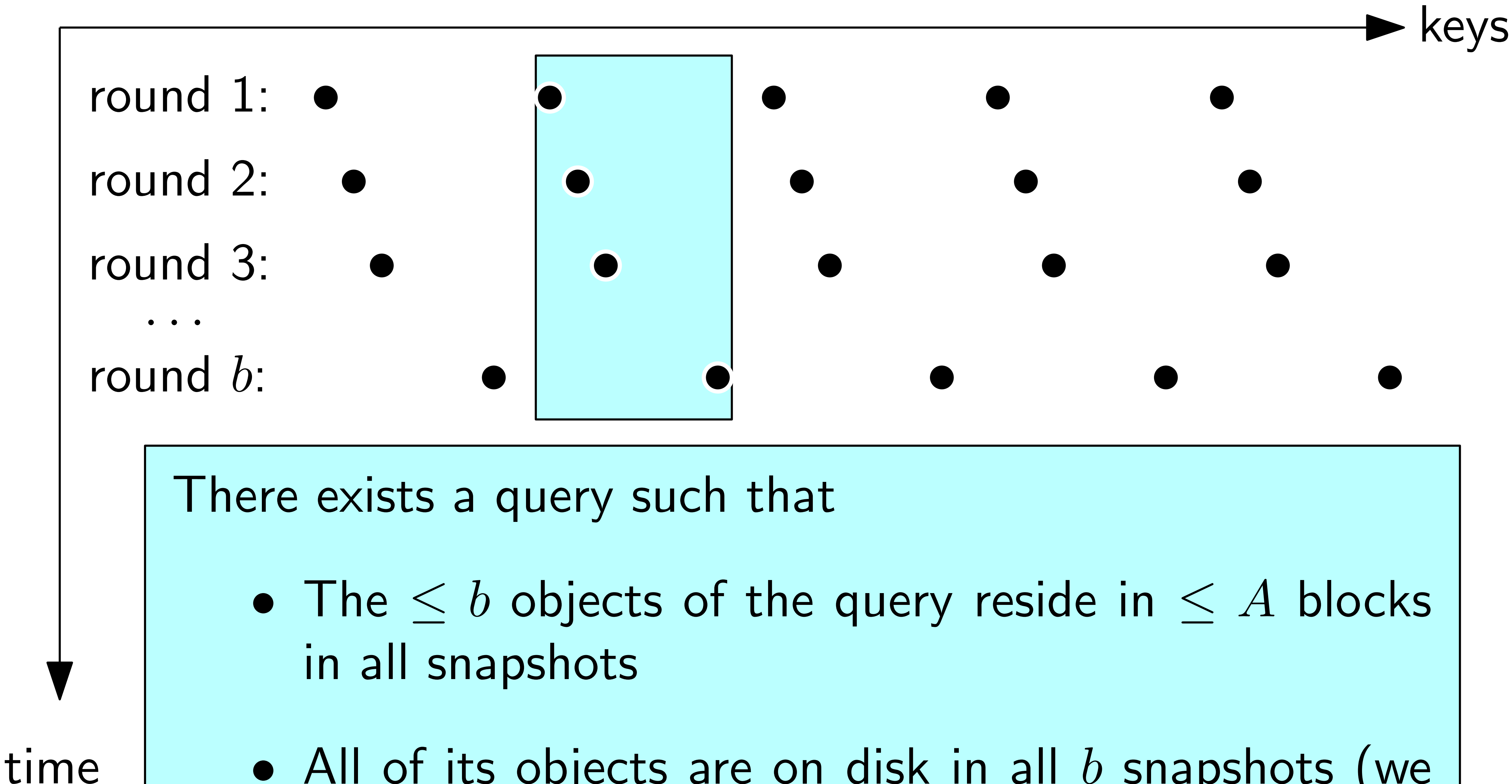
Snapshots of the dynamic index considered

The Workload Construction



There exists a query such that

- The $\leq b$ objects of the query reside in $\leq A$ blocks in all snapshots
- All of its objects are on disk in all b snapshots (we have $\geq mb$ queries)
- The index moves its objects ub times in total

The Reduction

An index with update cost u and access overhead A gives us a solution to the ball-shuffling game with cost ub for b balls and A bins

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$$\begin{cases} A \cdot \log(u/A) = \Omega(\log b), & \text{for } A < \alpha \ln b, \alpha \text{ is any constant;} \\ u \cdot \log A = \Omega(\log b), & \text{for all } A. \end{cases}$$

Ball-Shuffling Lower Bound Proof

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- Will show: Any algorithm that handles the balls with an average cost of u using A bins cannot accommodate $(2A)^{2u}$ balls or more.

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  - $u \to u + \frac{1}{2}$?

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- Need tol show: Any algorithm that handles the balls with an average cost of $u + \frac{1}{2}$ using A bins cannot accommodate $(2A)^{2u+1}$ balls or more.

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  To handle $(2A)^{2u+1}$ balls, any algorithm has to pay an average cost of more than $u + \frac{1}{2}$ per ball, or

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  - There are at least A good batches
  - Each good batch contributes at least $(2A)^{2u}$ to the "interference" cost

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- The recurrence

$$f_{A+1}(b) \geq \min_{k, x_1+\cdots+x_k=b} \{f_A(x_1-1)+\cdots+f_A(x_k-1) + kx_1+(k-1)x_2+\cdots+x_k-b\}$$

Open Problems and Conjectures

- 1D range reporting
  - Current lower bound: query $\Omega(\log b)$, update $\Omega(\frac{1}{b}\log b)$. Improve to $(\log\frac{n}{m},\ \frac{1}{b}\log\frac{n}{m})$?

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- Closely related problems: range sum (partial sum), predecessor search

The Grant Conjecture

	Internal memory (RAM) w: word size	External memory b: block size (in words)
range sum	$O : (\log n, \log n)$ binary tree $\Omega : (\log n, \log n)$ [Pătrașcu, Demaine, 06]	
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The End

THANK YOU

Q and A