The Reflex-Free Hull*

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We propose a hull operator, the *reflex-free hull*, that allows us to define a 3D analogue to bays in polygons. The reflex-free hull allows a rich set of topological types, yet for polyhedral input with n edges, it remains a polyhedral set with $O(n)$ edges. This is in contrast to other possible hull definitions that give non-planar surfaces and higher combinatorial complexity. The reflex-free hull is related to identifying cavities in computer aided design and manufacturing, but we sketch examples to indicate that computing a reflex-free hull will be a challenging problem.

1. Introduction

Computational geometers have identified many classes of 2D polygons (convex, star-shaped, L-convex, externally visible, edge-visible, LR-visible, street, person… [14,16]), but few classes of 3D polyhedra. Perhaps the fact that 3D polyhedra support rich classes of topological structure in the form of knots and links has overshadowed the identification of geometric structure.

In the plane, the difference between a simple polygon and its convex hull is a number of simple, polygonal bays, from which one can obtain a natural description of a polygon

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as a tree of unions and differences of convex pieces[15]. In space, it has been suggested that the same approach be used to define pockets in a search for casting directions[3], but in fact the difference between a polyhedron and its convex hull need not have a natural decomposition, and may have more complicated topology than the original polyhedron[1].

Feature recognition is an important research area in computer-aided design and manufacturing (CAD/CAM)[7,8]. Manufacturing features include geometric structures such as tunnels produced by drilling and cavities produced by milling or left behind by casting. Identifying these features from a solid model facilitates machining process planning or the analysis of manufacturability[7,11]. For example, cavities determine cast removal directions. Feature definitions in the CAD literature, however, are often restrictive to facilitate recognition. For example, Fields and Anderson[5] define a cavity as a connected set of facets with no convex edge, which means that the tiniest bump invalidates what would otherwise be a cavity.

Edelsbrunner et al.[4] defined *pockets* for α-*shapes*, special polyhedra that are induced by the unions of balls in three dimensions. Edelsbrunner et al. used pockets to model holes that are accessible through a narrow opening. In computational biology the union of balls models a macromolecule and pockets, which are cavities accessible through a narrow opening, model active sites that a foreign molecule may bind to. One way to extend the definition of pockets to general polyhedra might be to turn a given polyhedron P into an α-shape, but it is open whether there exists an α-shape for P whose combinatorial complexity is polynomial in the size of P.

In this paper, we suggest a definition of cavities that is analogous to the definition of bays in polygons. We define *reflex-free sets* and the *reflex-free hull* of a polyhedron P as an intersection of reflex-free sets containing it. The difference between P and its reflex-free hull gives the cavities. Section 2 provides the basic definitions. Section 3 describes a classification of points on the boundary of a three-dimensional closed bounded set. In Section 4, we use this classification to define reflex-free sets and the reflex-free hull, and we establish some of their properties. Section 5 proves that the reflex-free hull of a polyhedron has linear complexity even though it allows a rich set of topological types. Section 6 shows that the reflex-free hull is the limit of a process of filling cavities, but that obtaining it computationally in this manner would be challenging. Finally, Section 7 relates the reflex-free hull to other possible hull definitions, which either have high complexity or limited topologies.

2. Preliminaries

We present some basic geometric and topological concepts[2,9] in $\mathbb{R}^3$ that will be needed.

We use the Euclidean metric in $\mathbb{R}^3$. The distance between two points p and q is denoted by $d(p,q)$. For two subsets A and B of $\mathbb{R}^3$, the distance between them is $d(A,B) = \inf\{d(a,b) \mid a \in A, b \in B\}$. For any $\varepsilon > 0$, the *open* ε-*ball* centered at p is $B_\varepsilon(p) = \{q \in \mathbb{R}^3 \mid d(p,q) < \varepsilon\}$.

Let Q be a subset of $\mathbb{R}^3$. The *complement* $\overline{Q}$ of Q is defined as $\mathbb{R}^3 \setminus Q$. The set Q is *open* in $\mathbb{R}^3$ if for any point $p \in Q$, there exists $\varepsilon > 0$ such that $B_\varepsilon(p) \subset Q$. The set Q is *closed* in

$\mathbb{R}^3$ if $\overline{Q}$ is open. The empty set $\emptyset$, and its complement $\mathbb{R}^3$, are both open and closed. The *boundary* of Q, denoted by $\mathrm{bd}(Q)$, is the set $\{p \in \mathbb{R}^3 \mid \forall \varepsilon > 0, B_\varepsilon(p) \cap Q \neq \emptyset \wedge B_\varepsilon(p) \cap \overline{Q} \neq \emptyset\}$. Regardless of whether Q is open, closed, or neither, $\mathrm{bd}(Q)$ is always defined, although it may be empty. If Q is closed, then $\mathrm{bd}(Q) \subseteq Q$. If Q is open, then $\mathrm{bd}(Q) \cap Q = \emptyset$. The *closure* of Q, denoted by $\mathrm{cl}(Q)$, is defined as $Q \cup \mathrm{bd}(Q)$ and the *interior* of Q, denoted by $\mathrm{int}(Q)$, is defined as $Q \setminus \mathrm{bd}(Q)$. It can be readily checked that $\mathrm{cl}(Q)$ is closed and $\mathrm{int}(Q)$ is open. Since $\overline{\mathrm{bd}(Q)} = \mathrm{int}(Q) \cup \mathrm{int}(\overline{Q})$ which is open, $\mathrm{bd}(Q)$ is closed. For any point $p \in Q$, the ε-*neighborhood* of p is the intersection $B_\varepsilon(p) \cap Q$ for some $\varepsilon > 0$.

Two sets are *homeomorphic* if there is a continuous bijective map between them and the inverse map is also continuous. For any $0 \leq k \leq 3$, if every point of Q has a neighborhood homeomorphic to $\mathbb{R}^k$, then Q is a *k-manifold without boundary*. Otherwise, if every point of Q has a neighborhood homeomorphic to $\mathbb{R}^k$ or a closed halfspace in $\mathbb{R}^k$, then Q is a *k-manifold with boundary*. When Q is not a 3-manifold, we call a point of Q *singular* if none of its neighborhoods is homeomorphic to a $\mathbb{R}^3$ or a closed halfspace in $\mathbb{R}^3$.

A three-dimensional *simplicial complex* K is a collection of points, line segments, triangles, and tetrahedra. The convention is to include the empty set as an element of K too. Let S be the set of vertices of an element τ. Then $\partial\tau$ gives the collection of the convex hulls of all proper subsets of S. The elements of K are required to be closed under intersection and the operator ∂: (i) for any $\sigma, \tau \in K$, $\sigma \cap \tau \in K$, and (ii) for any $\tau \in K$, every $\sigma \in \partial\tau$ belongs to K. We call the underlying space of a finite simplicial complex (i.e., the union of its elements) a *polyhedral set*. If a polyhedral set is a 3-manifold with boundary, we also call it a *polyhedron*. There are polyhedral sets that are not polyhedra, for example, two tetrahedra joined at a single vertex (this vertex is singular) or along a single edge (each point on this edge is singular).

3. Classification of boundary points

Although 3-manifolds form a very general class of objects, we cannot restrict ourselves to them because a reflex-free hull may not be a 3-manifold in degenerate configurations. Let Q be a closed subset of $\mathbb{R}^3$. We classify each boundary point p of Q based intuitively on whether Q or $\mathrm{cl}(\overline{Q})$ can be oriented to hold water at p. Non-manifold sets complicate these definitions. We first define the appearances of the boundary points and local halfballs centered at them. Then we use these concepts to classify the boundary points.

Take any point $p \in \mathbb{R}^3$. For any vector v, we define the *v-plane* at p as $h_v(p) = \{q \in \mathbb{R}^3 \mid (q-p) \cdot v = 0\}$, and the closed *v-halfspace* at p as $h_v^-(p) = \{q \in \mathbb{R}^3 \mid (q-p) \cdot v \leq 0\}$. For any $\varepsilon > 0$ and any vector v, we define the (v,ε)-*halfball* at p as $H_{v,\varepsilon}(p) = B_\varepsilon(p) \cap h_v^-(p) \cap \overline{\{p\}}$. Note that $H_{v,\varepsilon}(p)$ is defined not to contain p so as to simplify the classification.

Suppose that p is a boundary point of Q. Take a connected component C of $B_\varepsilon(p) \cap \overline{Q}$ for some $\varepsilon > 0$. If $p \in \mathrm{cl}(C)$, this is an *appearance of p on Q* and $B_\varepsilon(p) \cap \overline{C}$ is the ε-*neighborhood of an appearance of p*. In contrast to the ε-neighborhood of p, the ε-neighborhood of an appearance of p needs not be a subset of Q. Since $C \subseteq \overline{Q}$, we have $B_\varepsilon(p) \cap Q \subseteq B_\varepsilon(p) \cap \overline{C}$, i.e., the ε-neighborhood of an appearance of p contains the ε-neighborhood of p. When p is singular, there may be one or more appearances of p; Fig. 1

shows some examples. Although p is non-singular in Fig. 1(a) and singular in Fig. 1(b), there is only one appearance of p in both cases. Fig. 1(c) shows the complementary case of Fig. 1(b). There are two appearances of p in Fig. 1(c) since $B_\varepsilon(p) \cap \overline{Q}$ consists of two open cones. The neighborhood of the appearance of p corresponding to the lower cone is shown in Fig. 1(d).

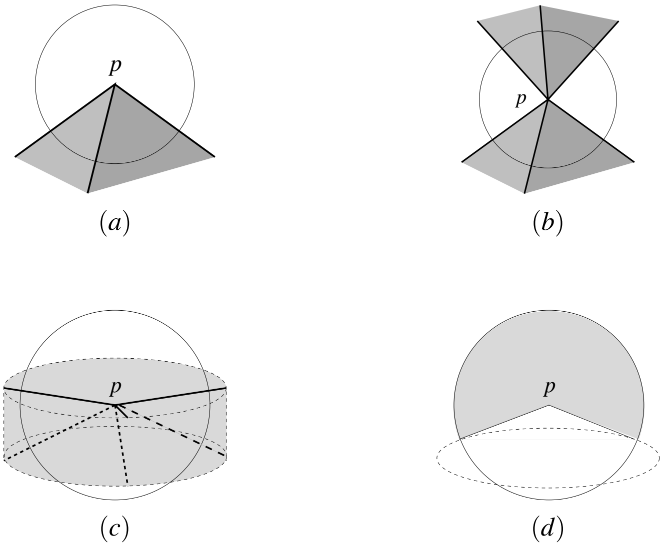


Fig. 1. The balls in the figures denote $B_\varepsilon(p)$ for some $\varepsilon > 0$. The shaded regions in (a), (b) and (c) denote Q in the vicinity of p. The shaded region in (d) denotes the ε-neighborhood of an appearance of p in (c).

Let C be a connected component in $B_\varepsilon(p) \cap \overline{Q}$ for some $\varepsilon > 0$. Let N denote $B_\varepsilon(p) \cap \overline{C}$; this is simply the neighborhood of p in Q when when p appears only once. We classify this appearance of p based on the relation of local halfballs to N. We say that this appearance of p is a

- *reflex point* if there is a vector v such that $H_{v,\varepsilon}(p) \subset \mathrm{int}(N)$.
- *convex point* if there is a vector v such that $H_{v,\varepsilon}(p) \subset \overline{N}$.
- *flat point* if there is a vector v such that $H_{v,\varepsilon}(p) \subset N$ and $H_{-v,\varepsilon}(p) \subset \mathrm{cl}(\overline{N})$.
- *nearly reflex point* if p is neither reflex nor flat, and there is a vector v such that $H_{v,\varepsilon}(p) \subset N$.
- *nearly convex point* if p is neither convex nor flat, and there is a vector v such that $H_{v,\varepsilon}(p) \subset \mathrm{cl}(\overline{N})$.
- saddle point otherwise. That is, this appearance of p is a saddle point if for any vector v, $H_{v,\varepsilon}(p) \cap \mathrm{int}(N) \neq \emptyset$ and $H_{v,\varepsilon}(p) \cap \overline{N} \neq \emptyset$.

Later, when we say that p appears with a particular point type, it means for some ε, there is an appearance of p of that point type. Similarly, when we say that Q has a point of a particular point type, it means for some ε, there is an appearance of some point of that point type.

Although some of our lemmas are general, we are primarily concerned with tame or even polyhedral sets in contrast to some of the "wild" sets defined in topology like Alexander's horned sphere.[9] Thus, we do not completely explore questions of classification for all sets. For example, suppose that a point p has an appearance C as a reflex point from some ε_0, i.e., C is a connected component in $B_{\varepsilon_0}(p) \cap \overline{Q}$ such that $p \in \mathrm{cl}(C)$. Suppose that this appearance of p is a reflex point. Then for any $\varepsilon < \varepsilon_0$, p also appears as a reflex point in every connected component C' in $C \cap B_\varepsilon(p)$ such that $p \in \mathrm{cl}(C')$—although the shape of the appearance changes as we go from C to C', the type of the appearance is preserved. The situation is more complicated for the other point types when p is singular. For example, if p appears in C as a convex point, then for some $\varepsilon < \varepsilon_0$, it is possible that $C \cap B_\varepsilon(p)$ consists of several components and p appears in one of them as a reflex point. Is there some ε_0 such that the classification of an appearance of p becomes stable for any $\varepsilon \leq \varepsilon_0$? Is any restriction on Q needed for this? We leave these as open questions.

When p is non-singular, there is only one appearance of p for any ε and so the classification of p is stable when ε is sufficiently small. Thus p has a unique point type. As an example, the boundary points of a polyhedron are classified as follows. The points inside a face are flat. If a dihedral angle of an edge is acute, then the points inside the edge are nearly reflex or nearly convex; otherwise, the points are flat. A vertex is a convex, reflex, or saddle point unless some of its incident edge/faces are coplanar. Fig. 2 illustrates the classification of points on a coffee mug. The reflex points lie at the rim of the bottom of the bowl.

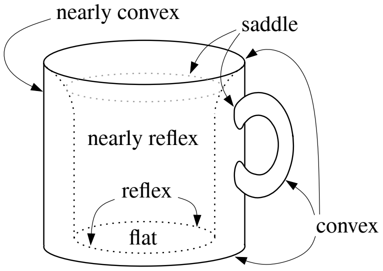


Fig. 2. Classifying boundary points

4. Reflex-free hull

We say that a set is a *reflex-free set* if and only if it is closed and no boundary point appears as a reflex point. The empty set and $\mathbb{R}^3$ are the smallest and largest reflex-free sets, respectively. Let Q be a closed subset of $\mathbb{R}^3$. Given a closed halfspace L, we call any bounded connected component in $\overline{Q} \cap L$ a *plane-cavity*. The next lemma gives an alternative definition of reflex-free sets using plane-cavities.

Lemma 1. *A closed subset Q of $\mathbb{R}^3$ has a plane-cavity if and only if Q has a reflex point.*

Proof. Suppose that a boundary point p of Q appears as a reflex point. Let N be the ε-neighborhood of an appearance of p and let $H_{v,\varepsilon}(p)$ be a halfball that demonstrates that p appears as a reflex point. Therefore, there is a connected component C in $B_\varepsilon(p) \cap \overline{Q}$ such that $p \in \mathrm{cl}(C)$, $N = B_\varepsilon(p) \cap \overline{C}$, and $H_{v,\varepsilon}(p) \subset \mathrm{int}(N)$. So C does not intersect $H_{v,\varepsilon}(p)$.

Let γ be a circle in the plane $h_v(p)$ with center p and radius $\varepsilon/2$. Because $h_v(p)$ bounds $H_{v,\varepsilon}(p)$, the circle $\gamma \subset H_{v,\varepsilon}(p) \subset \mathrm{int}(N)$. The idea is to shift the circle γ slightly so that the plane passing through the shifted circle cuts off a bounded connected component from $\overline{Q}$. We give the detailed arguments in the following.

Since $\gamma \subset \mathrm{int}(N)$, γ and $\mathrm{bd}(N)$ are disjoint. So the distance between γ and $\mathrm{bd}(N)$ is positive as both γ and $\mathrm{bd}(N)$ are compact, i.e., closed and bounded. Choose $\delta > 0$ to be less than the minimum of $\varepsilon/2$ and the distance between γ and $\mathrm{bd}(N)$. Let F be the family of circles $\{\gamma + \lambda v \mid \lambda \in [0, \delta]\}$. Let U be the union of disks bounded by the circles in F. Our choices of ε, δ, and the radius of γ imply that each circle in F lies inside $\mathrm{int}(N)$ and U lies inside $B_\varepsilon(p)$.

Every sufficiently small neighborhood of p lies inside $U \cup H_{v,\varepsilon}(p)$. This implies that U intersects C, since C does not intersect $H_{v,\varepsilon}(p)$. Also, C does not intersect the disk bounded by γ, since γ lies on $H_{v,\varepsilon}(p)$. Each circle in the family F lies inside $\mathrm{int}(N)$ and hence outside C. So any path in C between a point in $C \cap U$ and a point in $C \cap \overline{U}$ must pass through some interior point of the disk bounded by $\gamma + \delta v$. Moreover, since C is a connected component in $B_\varepsilon(p) \cap \overline{Q}$ and $B_\varepsilon(p)$ contains U, any boundary point of C in U must also be a boundary point of $\overline{Q}$. Hence the connected components in $C \cap U$ are bounded connected components in $\overline{Q} \cap h_v^-(p + \delta v)$. This completes the proof of the forward direction.

For the inverse, let X be a plane-cavity of Q. So X is a bounded connected component in $\overline{Q} \cap h_v^-(x)$ for some point $x \in \mathbb{R}^3$ and some vector v. Take a large sphere S that encloses $\mathrm{cl}(X)$ strictly inside. Let w be an outward normal vector of a face of $\mathrm{cl}(X)$ that lies on the plane $h_v(x)$. Move S in the direction of w until S touches $\mathrm{cl}(X)$. By making S sufficiently large, we can enforce that the contact point p between S and $\mathrm{cl}(X)$ does not lie on the plane $h_v(x)$. So p is a boundary point of Q. Choose $\varepsilon > 0$ to be less than the distance between p and $h_v(x)$. Let C be the connected component in $B_\varepsilon(p) \cap X$ such that $p \in \mathrm{cl}(C)$. Notice that C is a connected component in $B_\varepsilon(p) \cap \overline{Q}$ since $B_\varepsilon(p)$ avoids $h_v(x)$. Let u be a vector from p towards the center of S. The halfball $H_{u,\varepsilon}(p)$ lies outside S. Since $C \subseteq X$, which lies strictly inside S, $H_{u,\varepsilon}(p)$ is strictly contained in the interior of $B_\varepsilon(p) \cap \overline{C}$. It follows from definition that p appears as a reflex point. ☐

As a consequence, the intersection of reflex-free sets is reflex-free.

Lemma 2. *Let $\{Q_\alpha\}$ be a family of reflex-free sets. The intersection $\bigcap Q_\alpha$ is reflex-free.*

Proof. By the definition of reflex-free sets, Q_α is closed for every α. Thus $\bigcap Q_\alpha$ is closed. It remains to show that $\bigcap Q_\alpha$ has no reflex point which, by Lemma 1, is equivalent to showing that $\bigcap Q_\alpha$ has no plane-cavity. Assume to the contrary that there exists a closed halfspace L such that $\overline{(\bigcap Q_\alpha)} \cap L$ has a bounded connected component X. Let p be a point in X. Notice that the plane-cavities of $\bigcap Q_\alpha$ can be written as a union of individual plane cavities:

$$\overline{(\bigcap Q_\alpha)} \cap L = \bigcup (\overline{Q}_\alpha \cap L).$$

Since $p \in X \subseteq \bigcup(\overline{Q}_\alpha \cap L)$, p belongs to a connected component Y in $\overline{Q}_\alpha \cap L$ for some α. Since $Y \subseteq X$ and X is bounded, Y is bounded. But then Y is a plane-cavity of Q_α, contradicting the assumption that Q_α is reflex-free. ☐

For any closed subset Q of $\mathbb{R}^3$, define $Rfh(Q)$, the *reflex-free hull* of Q, as the intersection of all reflex-free sets that contain Q. For example, the reflex-free hull of a torus is the torus itself; the reflex-free hull of a coffee cup fills the cup but preserves the handle; the reflex-free hull of a set of discrete points is exactly the set of these points. The motivation is to find a structure that surrounds Q, but fills in the cavities. By Lemma 2, the reflex-free hull is indeed reflex-free. We show that the operator $Rfh()$ is idempotent.

Theorem 1. *For any closed subset Q of $\mathbb{R}^3$, the reflex-free hull $Rfh(Q)$ satisfies $Rfh(Rfh(Q)) = Rfh(Q)$.*

Proof. Since $Q \subseteq Rfh(Q)$, the sets whose intersection defines $Rfh(Rfh(Q))$ also participate in defining $Rfh(Q)$. Thus, it is clear that $Rfh(Q) \subseteq Rfh(Rfh(Q))$. We prove the reverse inclusion. By the definition of reflex-free hulls, if a point p is not in $Rfh(Q)$, then there is a reflex-free set R_p that contains Q but not p. Since $Q \subseteq R_p$ and R_p is reflex-free, $Rfh(Q) \subseteq R_p$ which implies that $Rfh(Rfh(Q)) \subseteq R_p$. Thus, p is not in $Rfh(Rfh(Q))$ as R_p does not include p. ☐

Next, we study two sculpting operations on reflex-free sets. They will be useful later in proving some structural properties of reflex-free hulls.

Lemma 3. *Let Q be a reflex-free set and let X be a closed set. If no reflex point of X lies in $\operatorname{int}(Q)$, the intersection $Q \cap X$ is reflex-free.*

Proof. First, $Q \cap X$ is a closed set. Assume to the contrary that a boundary point p of $Q \cap X$ appears as a reflex point. So there exists $\varepsilon > 0$, a vector v, and a connected component C in $B_\varepsilon(p) \cap \overline{(Q \cap X)}$ such that

- $p \in \mathrm{cl}(C)$, and
- $H_{v,\varepsilon}(p) \subset \operatorname{int}(N)$, where $N = B_\varepsilon(p) \cap \overline{C}$.

Observe that either $p \in \mathrm{bd}(Q)$ or $p \in \mathrm{bd}(X) \cap \operatorname{int}(Q)$. In the first case, we derive the contradiction that p appears as a reflex point of Q. It suffices to construct the ε-neighborhood of an appearance of p whose interior strictly contains $H_{v,\varepsilon}(p)$. Let $D = C \cap B_\varepsilon(p) \cap \overline{Q}$. Since C is connected component in $B_\varepsilon(p) \cap \overline{(Q \cap X)}$ and $B_\varepsilon(p) \cap \overline{Q} \subseteq B_\varepsilon(p) \cap \overline{(Q \cap X)}$, the set D is a connected component in $B_\varepsilon(p) \cap \overline{Q}$. The point p is in $\mathrm{cl}(D)$, since p is in $\mathrm{cl}(C)$. Thus, D demonstrates an appearance of p on Q. Notice that the neighborhood of the complement $M = B_\varepsilon(p) \cap \overline{D}$ contains N, since D is contained in C. It follows that $H_{v,\varepsilon}(p) \subset \mathrm{int}(N) \subset \mathrm{int}(M)$, so p appears as a reflex point, which is the desired contradiction.

In the second case, $p \in \mathrm{bd}(X) \cap \mathrm{int}(Q)$, we can substitute Q by X in the previous argument and show that p appears as a reflex point of X. But this contradicts the assumption that no reflex point of X lies in $\mathrm{int}(Q)$. $\square$

For any $\varepsilon > 0$, define the ε-*tube* for a line segment s as $C_\varepsilon(s) = \{x \in \mathbb{R}^3 \mid d(x,s) < \varepsilon\}$.

Corollary 1. *Let Y be a reflex-free set. If s is a line segment whose endpoints are at least $\varepsilon > 0$ from $\mathrm{cl}(Y)$, then $Y \setminus C_\varepsilon(s)$ is reflex-free.*

Proof. The condition on the endpoints of s ensures that the reflex points of $\overline{C_\varepsilon(s)}$ lie in $\overline{Y}$. Then Lemma 3 implies that $Y \setminus C_\varepsilon(s)$ is reflex-free. $\square$

5. The reflex-free hull of a polyhedron

The boundary of a polyhedral set can be partitioned into a set of polygonal faces (possibly with holes). We define the *size* of a polyhedral set to be the number of vertices, edges, and faces on its boundary. The main result of this section is that the reflex-free hull of a polyhedron is a polyhedral set of the same asymptotic size.

We extend the definition of open sets to subspaces of $\mathbb{R}^3$. For $1 \leq k \leq 3$, a set S is an *open set with dimension k* if there is a k-manifold without boundary M such that for any point $p \in S$, $B_\varepsilon(p) \cap M \subset S$ for some $\varepsilon > 0$. A suitable manifold is usually obvious. For example, a line for an open line segment; a plane for an open disk. By restricting to points in M (i.e., replacing $\mathbb{R}^3$ by M and $B_\varepsilon(p)$ by $B_\varepsilon(p) \cap M$), we can define closed sets in M, boundary, closure, and interior as before. For example, the boundary of an open line segment consists of its endpoints; the boundary of an open disk is the bounding circle.

Let P denote our polyhedron and let n denote its size. Intuitively, the non-singular flat points in $\mathrm{bd}(\mathit{Rfh}(P))$ form a set of polygons that cover $\mathrm{bd}(\mathit{Rfh}(P))$. To this end, we need to argue that the singular points and the non-flat points form a finite number of vertices and edges. In Section 5.1, we show that the non-singular nearly reflex points form a family of open line segments. In Section 5.2, we use this result to prove that $\mathit{Rfh}(P)$ is a polyhedral set with $O(n)$ vertices. In Section 5.3, we finish the proof of the main result by bounding the number of edges and faces. The following technical result will be useful.

Lemma 4. *Let ab be a line segment. If $a,b \in \overline{\mathit{Rfh}(P)}$ and ab intersects $\mathit{Rfh}(P)$, then ab intersects P.*

Proof. Note that both P and $\mathit{Rfh}(P)$ are closed. Since $\overline{\mathit{Rfh}(P)}$ is therefore open, we can perturb a given a and b to points $a', b' \in \overline{\mathit{Rfh}(P)}$, so that $a'b'$ intersects the interior $\mathrm{int}(\mathit{Rfh}(P))$. If ab does not intersect P, we can make the perturbation sufficiently small that $a'b'$ avoids P. But then by Corollary 1, for some $\varepsilon > 0$, $\mathit{Rfh}(P) \setminus C_\varepsilon(a'b')$ is a smaller reflex-free set containing P, a contradiction. $\square$

5.1. *Nearly reflex points*

Let NR be the set of non-singular nearly reflex points of $Rfh(P)$ that are not vertices of P and do not lie on any convex edge of P. Our goal is to show that NR is a family of open line segments. We first examine the surrounding of certain boundary points of $Rfh(P)$.

Lemma 5. *Let p be a boundary point of $Rfh(P)$. Suppose that p is not a vertex of P and p does not lie on any convex edge of P. There exists $\delta > 0$ such that for all $\varepsilon < \delta$,*

(i) $B_\varepsilon(p)$ intersects only the edges and faces of P that p is incident on,
(ii) $B_\varepsilon(p) \setminus Rfh(P)$ is convex, and
(iii) If $B_\varepsilon(p) \cap Rfh(P)$ is non-empty, it is homeomorphic to a closed halfspace in $\mathbb{R}^3$.

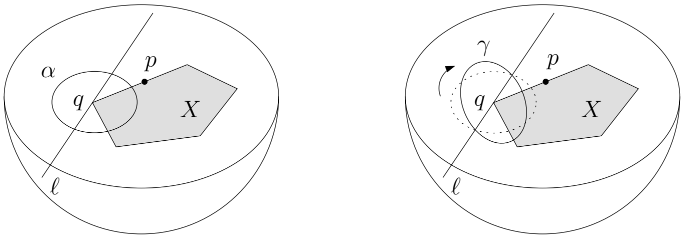
Proof. Since P is a polyhedron, it is trivial to find δ that satisfies (i). For (ii), let $X = B_\varepsilon(p) \setminus Rfh(P)$ for any $\varepsilon < \delta$. Assume to the contrary that X is non-convex, so there is a line segment s with endpoints in X such that s intersects $Rfh(P)$. By Lemma 4, s also intersects P. Since the endpoints of s are outside P, p is a vertex of P or p lies on a convex edge of P, contradicting the assumption of the lemma. So (ii) is also satisfied.

For (iii), pick any point $x \in B_\delta(p) \setminus Rfh(P)$. Orient space so that x is vertically above p. There exists $\alpha > 0$ so that $B_\alpha(x) \subset B_\delta(p) \setminus Rfh(P)$. Take any $\varepsilon < \alpha$ so that $B_\varepsilon(p)$ and $B_\alpha(x)$ are disjoint. The set $Y = B_\varepsilon(p) \setminus Rfh(P)$ is convex by (ii). So the lower envelope of Y is a topological disk. By convexity, $B_\delta(p) \setminus Rfh(P)$ contains the convex hull of $B_\alpha(x)$ and Y. It follows that $B_\varepsilon(p) \cap \mathrm{bd}(Y)$ is the lower envelope of Y, since $B_\alpha(x)$ is above the entire Y. So $B_\varepsilon(p) \cap \mathrm{bd}(Rfh(P)) = B_\varepsilon(p) \cap \mathrm{bd}(Y)$ is an open topological disk, i.e., (iii) is satisfied. In all, it suffices to reduce δ until $B_\delta(p)$ and $B_\alpha(x)$ are disjoint. ☐

Next, we show that each nearly-reflex points can be organized into line segments. Consider a point $p \in NR$, and let L_p be the collection of maximal open line segments, each consisting of points of NR, that either go through p or have p as a boundary point.

Lemma 6. *For every point $p \in NR$ and sufficiently small $\varepsilon > 0$, the neighborhood $B_\varepsilon(p) \cap L_p$ contains an open line segment that has p in its interior.*

Proof. Choose $\varepsilon > 0$ to satisfy Lemma 5 for p. Since $B_\varepsilon(p) \setminus Rfh(P)$ is convex by Lemma 5(ii), there exists a vector v such that halfball $H_{v,\varepsilon}(p) \subset B_\varepsilon(p) \cap Rfh(P)$. Let D be the equatorial disk of $B_\varepsilon(p)$ that bounds $H_{v,\varepsilon}(p)$. The set $X = D \cap \mathrm{cl}(B_\varepsilon(p) \setminus Rfh(P))$ is convex, since D and $B_\varepsilon(p) \setminus Rfh(P)$ are convex. Notice that X is equal to $D \cap \mathrm{bd}(Rfh(P))$. So $D \setminus X \subset \mathrm{int}(Rfh(P))$. Since p is non-singular and nearly reflex, any small disk on D centered at p intersects both $\mathrm{int}(Rfh(P))$ and $\mathrm{bd}(Rfh(P))$. Thus $X \setminus \{p\}$ is non-empty and p lies on the boundary of X. We claim that no line that is coplanar with D touches X locally at a single point. Assume to the contrary that such a line ℓ exists. Let q be the contact point between ℓ and X. Let α be an arbitrarily small circle on D centered at q, as in Fig. 3(a). Since X is convex, we can apply the same arguments in proving Lemma 5(iii) to show that $\alpha \setminus X$ is a contiguous arc. Note that $\alpha \setminus X \subset D \setminus X \subset \mathrm{int}(Rfh(P))$. We form circle γ by rotating α around ℓ so that the semicircle that was in X rotates away from $H_{v,\varepsilon}(p)$,



(a) (b)

Fig. 3. Form γ by rotating circle α about line l on the boundary of halfballs $H_{v,\varepsilon}(p)$ in the proof of Lemma 6.

as illustrated in Fig. 3(b). We make the rotation angle sufficiently small that γ lies inside $B_\varepsilon(p) \cap \mathrm{int}(Rfh(P))$. The disk bounded by γ contains the point q although it may not touch $\mathrm{bd}(Rfh(P))$ at q. In any case, we can shift γ as in the proof of Lemma 1 to cut off a plane-cavity from $B_\varepsilon(p) \setminus Rfh(P)$. But then Lemma 1 implies that $Rfh(P)$ has a reflex point, a contradiction.

Our claim implies that the boundary of X contains a diameter of D. We observe that any point on this diameter is nearly reflex, because we can shrink $H_{v,\varepsilon}(p)$ to a small halfball centered at that point. Thus this diameter is an open line segment that satisfies the lemma. ☐

Next, we strengthen Lemma 6 and show that L_p contains a single line.

Lemma 7. *For each point $p \in NR$, the set L_p consists of exactly one open line segment that contains p in its interior.*

Proof. By Lemma 6, there is an open line segment ℓ_1 in L_p that contains p in its interior. If the lemma is false, there is another open line segment ℓ_2 in L_p. Choose $\varepsilon > 0$ sufficiently small that the only edges and faces of P that $B_\varepsilon(p)$ intersects are those incident to p. The line segment ℓ_2 intersects $B_\varepsilon(p)$, since p lies on ℓ_2 or p is a boundary point of ℓ_2. Pick a point q on $B_\varepsilon(p) \cap \ell_2$ other than p. Pick two points r_1 and r_2, one from each side of p on $B_\varepsilon(p) \cap \ell_1$. Any halfball showing that q is nearly reflex in $Rfh(P)$ is in tangential contact with ℓ_2. Also, any small disk that lies on this halfball and is centered at q intersects $\mathrm{int}(Rfh(P))$. Since qr_1, qr_2, and ℓ_2 are coplanar, either both line segments qr_1 and qr_2 lie on this halfball or one of them stabs through it. Without loss of generality, we may assume that qr_1 intersects $\mathrm{int}(Rfh(P))$. We perturb q and r_1 to points $q', r'_1 \in B_\varepsilon(p) \setminus Rfh(P)$ so that $q'r'_1$ still intersects $\mathrm{int}(Rfh(P))$. By Lemma 4, $q'r'_1$ intersects P, which implies that p is a vertex of P or p lies on a convex edge of P, contradicting the assumption that $p \in NR$. ☐

We have shown that NR is a family of disjoint open line segments. For every point $p \in NR$, we will simply use L_p to denote the open line segment containing it.

5.2. *Polyhedral set*

We show that $\mathit{Rfh}(P)$ is a polyhedral set in this section. Clearly, the non-singular flat points group into polygons. So we need to prove that boundary points of other types form a finite number of vertices and edges. The next lemma studies how the singular, convex, nearly convex, and saddle points of $\mathit{Rfh}(P)$ are inherited from P.

Lemma 8. *If p is a singular point of $\mathit{Rfh}(P)$, then p is a vertex of P or p lies on a convex edge of P. Otherwise,*

(i) if p is a convex point of $\mathit{Rfh}(P)$, p is a convex point of P;
(ii) if p is a nearly convex point of $\mathit{Rfh}(P)$, p is a convex or nearly convex point of P;
(iii) if p is a saddle point of $\mathit{Rfh}(P)$, p is a convex, nearly convex or saddle point of P.

Proof. If p is singular, then no neighborhood of p is homeomorphic to a closed halfspace in $\mathbb{R}^3$. The contrapositive of Lemma 5(iii) implies that p is a vertex of P or p lies on a convex edge of P. If p is non-singular and a convex, nearly convex, or saddle point of $\mathit{Rfh}(P)$, then observe that $p \in \mathrm{bd}(P)$. Otherwise, by Lemma 5(ii), there exists a vector v and $\varepsilon > 0$ such that $H_{v,\varepsilon}(p) \subset B_\varepsilon(p) \cap \mathit{Rfh}(P)$, contradicting the assumption that p is convex, nearly convex, or saddle. Claims (i), (ii), and (iii) follow by checking the local halfballs at p against $\mathit{Rfh}(P)$ and P. □

Not only are the nearly convex points of $\mathit{Rfh}(P)$ inherited from P, but any convex edge of P that appears on $\mathit{Rfh}(P)$ has a single interval where it appears nearly convex or flat.

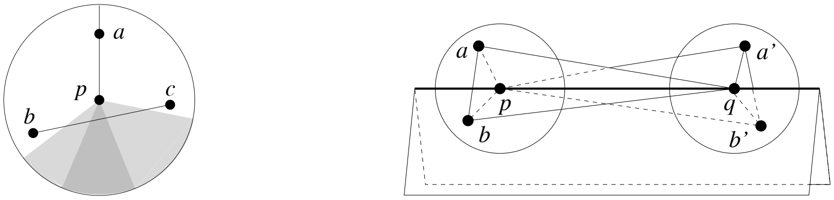
Lemma 9. *Let p and q be two points from the interior of a convex edge of P that appear on $\mathrm{bd}(\mathit{Rfh}(P))$, but not as nearly reflex points. Then all points in $\mathrm{int}(pq)$ are non-singular boundary points of $\mathit{Rfh}(P)$, and either all of them are flat or all of them are nearly convex.*

Proof. Let p and q be points satisfying the hypothesis of the lemma. Let e be the convex edge of P that contains p and q. Choose $\varepsilon > 0$ so that $B_\varepsilon(p)$ and $B_\varepsilon(q)$ are disjoint, and that the tube $C_\varepsilon(pq)$ intersects only the two faces of P incident to e. Let G be the plane through e that bisects $B_\varepsilon(p) \cap P$ and $B_\varepsilon(q) \cap P$.

We claim that we can find a point $a \in B_\varepsilon(p) \setminus \mathit{Rfh}(P)$ and two points $b, c \in B_\varepsilon(p) \setminus \mathrm{int}(\mathit{Rfh}(P))$ such that a lies on G and bc intersects P. See Fig. 4(a). There are two cases to consider.

Case 1: $B_\varepsilon(p) \setminus \mathit{Rfh}(P)$ is convex. By convexity, there exists a vector v such that $H_{v,\varepsilon}(p) \subset B_\varepsilon(p) \cap \mathit{Rfh}(P)$. So p appears as reflex, nearly reflex, or flat. $\mathit{Rfh}(P)$ has no reflex point and p does not appear as nearly reflex by assumption. So p is flat. The points b and c can be easily picked from the bounding disk of $H_{v,\varepsilon}(p)$. The point a can be any point in $B_\varepsilon(p) \cap \overline{\mathit{Rfh}(P)} \cap G$ near p.

Case 2: $B_\varepsilon(p) \setminus \mathit{Rfh}(P)$ is non-convex. There is a line segment bc such that $b, c \in B_\varepsilon(p) \setminus \mathit{Rfh}(P)$ and bc intersects $\mathit{Rfh}(P)$. By Lemma 4, bc intersects P. So b and c lie on opposite sides of G. The plane through b and the edge e cuts off a halfball $H_{v,\varepsilon}(p)$ from $B_\varepsilon(p)$ that avoids P. Since p does not appear as nearly reflex, the interior



(*a*) (*b*)

Fig. 4. The balls denote $B_\varepsilon(p)$ and $B_\varepsilon(q)$. Figure (a) shows the view along the convex edge e of P. The smaller and larger shaded regions denote $B_\varepsilon(p) \cap P$ and $B_\varepsilon(p) \cap Rfh(P)$, respectively. Note that the points a, b, c and p need not be coplanar. Figure (b) shows the two faces of P incident to e and the two tetrahedra τ and τ'. The points a, a', p and q lie on the same vertical plane.

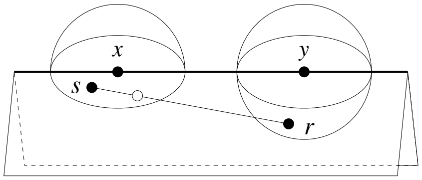


Fig. 5. The figure shows the two faces of P incident to the convex edge e. The white dot is the intersection point between rs and $h_v(x)$.

> of $H_{v,\varepsilon}(p)$ contains a point x in $\overline{Rfh(P)}$. Either bx or cx intersects G, say cx. If cx avoids P, the contrapositive of Lemma 4 implies that $bx \subset \overline{Rfh(P)}$. So we set $a = cx \cap G$ to satisfy the claim. If cx intersects P, we replace b by x and repeat. Notice that the dihedral angle between the triangles formed by b and c with e strictly increases. So this process must terminate.

Similarly, we can find a point $a' \in B_\varepsilon(q) \setminus Rfh(P)$ and two points $b', c' \in B_\varepsilon(q) \setminus \mathrm{int}(Rfh(P))$ such that a' lies on G and $b'c'$ intersects P.

We can pick a point from $\{b, c\}$ and a point from $\{b', c'\}$, say b and b', so that they lie on opposite sides of G and the dihedral angle between the triangles bpq and $b'pq$ is at most π. See Fig. 4(b). Let τ and τ' be the tetrahedra $abpq$ and $a'b'pq$, respectively. Notice that τ and τ' do not intersect $\mathrm{int}(P)$. By Lemma 3, $Rfh(P)$ must avoid $\overline{\mathrm{int}(\tau \cup \tau')}$ so that no further sculpting is possible. So $\mathrm{int}(pq) \subset \mathrm{bd}(Rfh(P))$ and each point in $\mathrm{int}(pq)$ appears as flat or nearly convex. It remains to argue that points in $\mathrm{int}(pq)$ are non-singular and they have the same point type. Orient space so that the plane G is vertical, e is horizontal, and $\tau \cup \tau'$ lies above e. Let v be the vector pointing vertically downward.

Let x be a point in $\mathrm{int}(pq)$. Choose $\delta > 0$ so that $H_{v,\delta}(x) \subset \tau \cup \tau'$. We claim that $B_\delta(x) \cap Rfh(P)$ is vertically monotone. Otherwise, there are two vertical intervals $I_1, I_2 \subset B_\delta(x) \cap Rfh(P)$ such that $\tau \cup \tau'$ is above I_1, and I_1 is directly above a point in $\overline{Rfh(P)}$ which is directly above I_2. But then Lemma 4 implies that I_1 intersects P, which is impossible since

I_1 is above a point in $\overline{Rfh(P)} \subseteq \overline{P}$. By the vertical monotonicity, a small neighborhood of x in $Rfh(P)$ that lies entirely above $B_\delta(x) \cap P$ is homeomorphic to a closed halfspace in $\mathbb{R}^3$. So x is non-singular.

Assume to the contrary that there are two points $x, y \in \text{int}(pq)$ such that x is flat and y is nearly convex. Choose $\delta > 0$ so that $H_{v,\delta}(x)$ and $H_{v,\delta}(y)$ lie inside $\tau \cup \tau'$. Since x is flat, the bounding disk of $H_{v,\delta}(x)$ contains a disk $D \subset \text{bd}(Rfh(P))$ centered at x. Since y is nearly convex, $B_\delta(y) \setminus H_{v,\delta}(y)$ contains a point r in $\overline{Rfh(P)}$. See Fig. 5. Draw a line segment from r, stabbing the disk D, to a point s in the interior of $H_{v,\delta}(x)$. This can be done so that rs avoids P. But as rs stabs $D \subset Rfh(P)$, rs intersects P by Lemma 4, a contradiction. □

In the worst case, a convex edge e of P contributes at most three consecutive intervals to $\text{bd}(Rfh(P))$: points in the two extreme intervals appear as nearly reflex, and points in the middle interval are flat or nearly convex, according to Lemma 9. We claim that each nearly reflex or nearly convex interval forms a single edge, i.e., no other edge has an endpoint inside it. By Lemma 8, the convex, nearly convex, and saddle points of $Rfh(P)$ not on e do not form any edge with an endpoint inside e. It remains to deal with the points in NR. The next lemma addresses this.

Lemma 10. *Let e be a convex edge of P. If two points $x, y \in e$ are in* $\text{bd}(Rfh(P))$*, then* $\text{int}(xy)$ *does not contain any boundary point of L_p for all points $p \in NR$.*

Proof. Assume that for some point $p \in NR$, line segment L_p has a boundary point q in $\text{int}(xy)$. Choose $\varepsilon > 0$ so that the tube $C_\varepsilon(xy)$ intersects only the two faces of P incident to e. Pick a point $z \in B_\varepsilon(q) \cap L_p$ other than q. Any halfball showing that z is nearly reflex in $Rfh(P)$ is in tangential contact with L_p. Also, any small disk, that lies on this halfball and is centered at z, intersects $\text{int}(Rfh(P))$. Since xz, yz, and L_p are coplanar, either xz and yz lie on this halfball or one of them stabs through it. In any case, xz or yz intersects $\text{int}(Rfh(P))$, say xz. We can pick two points $a, b \in B_\varepsilon(q) \setminus Rfh(P)$ near z such that the tetrahedron $\tau = abxz$ intersects P only at points on xz. By Lemma 3, $Rfh(P) \cap \overline{\text{int}(\tau)}$ is a smaller reflex-free set containing P, a contradiction. □

Next, we prove that $Rfh(P)$ is a polyhedral set and bound the number of its vertices.

Lemma 11. *The reflex-free hull of a polyhedron of size n is a polyhedral set with $O(n)$ vertices.*

Proof. Let P be a polyhedron of size n. We show that the boundary points of $Rfh(P)$ that are singular or non-flat form a finite number of vertices and edges.

Let R be the set of boundary points of $Rfh(P)$ that appear as nearly reflex points. We consider the subsets $R \setminus NR$ and NR separately. By the definition of NR, there are two options for a point $x \in R \setminus NR$:

(i) x is singular;
(ii) x is a vertex of P or it lies on a convex edge of P.

By Lemma 8, we know that (i) reduces to (ii). By Lemmas 9 and 10, the points in $R \setminus NR$ on each convex edge of P form at most two edges. Thus the points in $R \setminus NR$ form $O(n)$ vertices and $O(n)$ edges. We know that NR consists of open line segments that are non-crossing and on one contains a boundary point of another. For each point $p \in NR$, there are several options for a boundary point x of L_p:

(i) x is non-singular and not nearly reflex;
(ii) x is singular;
(iii) x is a vertex of P or it lies on a convex edge of P.

For (i), x is flat, convex, nearly convex, or saddle. But x cannot be flat because the neighborhood of a flat point is incompatible with the neighborhood of nearly reflex points on L_p that converge to x. Then Lemma 8 implies that (i) and (ii) reduce to (iii). By Lemma 10, only the two extreme points in $\mathrm{bd}(Rfh(P))$ on a convex edge of P can be incident on open line segments in NR. Hence, there are $O(n)$ possible boundary points for the open line segments in NR, which implies that NR consists of $O(n^2)$ open line segments. In all, R has $O(n)$ vertices and $O(n^2)$ edges.

By Lemma 9, the interior of each convex edge of P contains at most two singular points of $Rfh(P)$ that are not in R. Then Lemma 8 implies that there are $O(n)$ singular points of $Rfh(P)$ that are not in R. Similar reasoning shows that there are $O(n)$ convex and saddle points of $Rfh(P)$. By Lemmas 9 and 10, the nearly convex points of $Rfh(P)$ on a convex edge of P form at most one edge. Then Lemma 8 implies that the nearly convex points of $Rfh(P)$ form $O(n)$ vertices and $O(n)$ edges.

In all, the boundary points of $Rfh(P)$ that are singular or non-flat points form $O(n)$ vertices and $O(n^2)$ edges. Since the number of vertices and edges is finite, they must form the boundaries of the polygons comprised of the non-singular flat points. It follows that $Rfh(P)$ is a polyhedral set with $O(n)$ vertices. ☐

5.3. *The size of the reflex-free hull*

We are ready to prove the main result of this section: for a polyhedron P of size n, the reflex-free hull $Rfh(P)$ is a polyhedral set of size $O(n)$. In Lemma 11, we already showed that there are $O(n)$ vertices, but we only proved a loose $O(n^2)$ bound on the number of edges. We tighten this bound by analyzing the genus of $Rfh(P)$.

Theorem 2. *The reflex-free hull of a polyhedron of size n is a polyhedral set of $O(n)$ size.*

Proof. From Lemma 11 we know that $Rfh(P)$ is a polyhedral set with $O(n)$ vertices. We only have to bound the number of faces and edges.

The Euler-Poincaré formula (due to Poincaré 1899) states that $V - E + F = 2 - 2g$, where V, E, and F are the numbers of vertices, edges, and faces of $Rfh(P)$, respectively, and g is its genus. Since $3F \leq 2E$, we deduce that $V - E/3 \geq 2 - 2g$ and so $E \leq 3V + 6g - 6$. Since V is $O(n)$, we only have to show that $g = O(n)$ to complete the proof.

Curvature is defined at all points on a surface, and from the Gauss-Bonnet theorem[6] we know that the sum of curvature over $Rfh(P)$ equals $-4\pi(g-1)$. For a polyhedron, the points on faces or edges have zero curvature, and the curvature of a vertex equals 2π minus the sum of the angles of its incident faces. Thus, $4\pi g$ equals 2π plus the sum of the face angles at all vertices of $Rfh(P)$. The genus of P is less than n by the Euler-Poincare formula. We bound the increase in genus from P to $Rfh(P)$ by bounding the increase in the sum of face angles at vertices. There are three types of changes to vertices when we go from P to $Rfh(P)$:

(i) a vertex v of P may disappear into the interior of $Rfh(P)$,
(ii) a new vertex v may be created on $Rfh(P)$, or
(iii) a vertex v of P may become incident to new faces.

We bound the corresponding increase in the sum of face angles as follows.

(i) When a vertex v of P disappears, it no longer contributes to the sum of face angles.

(ii) A new vertex v lies in the interior of a convex edge e of P. By Lemma 9, at most one convex edge of $Rfh(P)$ lies on e, which is the only possible convex edge incident to v because other convex edges of $Rfh(P)$ lie on convex edges of P that are disjoint from $\mathrm{int}(e)$. If v is incident to a convex edge, the angles of the two faces incident to this edge sum to 2π or less. Since the other incident edges of v are reflex, the sum of the remaining face angles is less than 2π. So a new vertex increases the sum of face angles by less than 4π.

(iii) Since $Rfh(P)$ is a polyhedral set, we can organize the faces incident to v into one or more topological disks. At most one of these disks does not contain a convex edge. Such a disk must be flat as v cannot be a reflex vertex. So the sum of face angles on it is 2π.

Let F_{new} be a topological disk of incident faces of v in $Rfh(P)$ that contains a convex edge. Let F_{old} be the topological disk of incident faces of v in P. Let S be a sphere of directions centered at v that intersects the incident faces of v in $Rfh(P)$ and P only. F_{old} (resp., F_{new}) cuts out a region D_{old} (resp., D_{new}) from S that lies inside P (resp., $Rfh(P)$).

By Lemma 8, the convex edges of $Rfh(P)$ incident to v are parts of convex edges of P incident to v. The change from P to $Rfh(P)$ replaces the faces between two convex edges (possibly identical) by a new set of faces that are joined by reflex edges. It follows that $D_{\text{old}} \subseteq D_{\text{new}}$, and the boundaries of D_{old} and D_{new} touch at one or more points. Since P is a polyhedron, $D_{\text{old}} = S \cap P$ which is connected. As $D_{\text{old}} \subseteq D_{\text{new}}$, F_{new} is the only topological disk of incident faces of v in $Rfh(P)$ that contains a convex edge.

The set $D_{\text{new}} \setminus D_{\text{old}}$ consists of several disjoint regions. Let $R(\eta, \sigma)$ be one such region, where η and σ are its boundary curves that come from D_{new} and D_{old}, respectively. Let A_η and A_σ denote the sums of angles of faces traversed by η and σ, respectively. Clearly, $A_\eta \leq 2\pi$ as all edges traversed by η are reflex. We conduct an analysis below to relate A_η and A_σ. All angles are measured in radian.

Let p and q denote the common endpoints of η and σ. If $\angle pvq \geq 1$, then $A_\sigma \geq \angle pvq \geq 1$ and so $A_\eta \leq 2\pi A_\sigma$. Suppose that $\angle pvq < 1$. Let S_{pq} be the cap

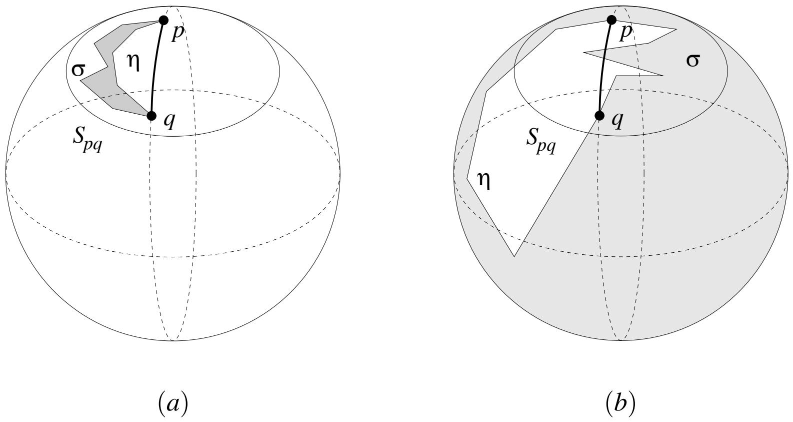


Fig. 6. The big spheres denote S. The shaded regions denote $R(\eta,\sigma)$. In figure (b), $R(\eta,\sigma)$ contains the quarter sphere at the bottom right.

on S with angular radius 1 radian and centered at the midpoint of the geodesic connecting p and q. If σ leaves S_{pq}, then $A_\sigma \geq 1$ because the angular distances from p and q to the boundary of S_{pq} are at least 0.5 radian. So $A_\eta \leq 2\pi A_\sigma$. Suppose that σ lies inside S_{pq}. See Fig. 6. If the region $R(\eta,\sigma)$ lies inside S_{pq}, then since all edges traversed by η are reflex, $A_\eta \leq A_\sigma$. See Fig. 6(a). If $R(\eta,\sigma)$ does not lie inside S_{pq}, then although η may or may not lie inside S_{pq}, η cannot cross the equatorial circle through p and q. See Fig. 6(b). Therefore, $R(\eta,\sigma)$ contains at least one-quarter of S. The last case cannot happen more than four times among all disjoint regions in $D_{\text{new}} \setminus D_{\text{old}}$. In all, the sum of face angles in F_{new} is at most 8π plus 2π times the sum of face angles in F_{old}.

Summing the increase in the sum of face angles in (ii) and (iii) over all vertices of $Rfh(P)$, we conclude that the genus of $Rfh(P)$ is $O(\text{genus of } P) + O(n)$, which is $O(n)$. This completes the proof of Theorem 2. □

6. The reflex-free hull and cavities

Recall that for a closed polyhedral set Q and halfspace h^-, we defined a *plane-cavity* as a connected component of $\overline{Q} \cap h^-$. We can enlarge a plane-cavity by translating its plane h unless h contains a saddle point or nearly convex points. We say that a plane-cavity is *limited* if its plane contains three saddle points or a closed curve of convex and nearly convex points.

Lemma 12. *Any plane-cavity is contained in the union of four limited plane-cavities.*

Proof. Consider a plane-cavity $C \subset \overline{Q} \cap h^-$. We may translate h to enlarge C unless doing so would cause C to be connected to the unbounded component of $\overline{Q}$. This may happen if h contains a closed chain of (nearly) convex points satisfying the lemma.

Otherwise h contains one or more saddle points. If h contains two saddle points, a and b, then we may duplicate h and rotate the two copies in opposite directions around the line ab. We stop each rotation when a third saddle point or chain of nearly convex points is reached. If there is a single saddle, we again duplicate and rotate h around some line through the saddle until we hit a second saddle and reduce to two instances of the previous case. □

If we iteratively fill up plane cavities for a polyhedron P, then we obtain a sequence of interesting sets. We describe this process precisely as follows. Let $P_0 = P$. Given some plane-cavity C_k of P_k, we form the union $P_k \cup C_k$ to obtain a new polyhedron P_{k+1}. We may choose our plane-cavities by always choosing the one with largest volume, or by always choosing four limited plane-cavities that enclose the largest volume.

We call a connected component of $P_k \setminus P$ a *cavity*. We believe, but have not been able to formally prove, that in the limit we can obtain the reflex-free hull by filling cavities. Equivalently the cavities of a closed polyhedron P are the connected components of $Rfh(P) \setminus P$.

Theorem 3. *For a closed polyhedron P, the limit of the process of filling cavities is a subset of the reflex-free hull,* $Rfh(P)$.

Proof. We show by induction that the cavities identified by the filling process are inside all reflex-free sets that contain P. Specifically, we prove that any polyhedral set Q that contains P, but does not contain P_k, has a reflex point.

The base case, $k = 0$, is trivial; no set containing P can omit a point of $P_0 = P$.

For the inductive step, we assume the induction hypothesis for some $k \geq 0$, and prove it for $k+1$. Thus, assume that Q is a polyhedral set that contains P but does not contain P_{k+1}. If Q does not contain P_k, then the induction hypothesis applies immediately, so we assume that Q contains P_k. The boundary of Q therefore intersects the plane-cavity $C_k = P_{k+1} \setminus P_k$. But since $\mathrm{bd}(Q)$ can only escape through the plane defining C_k, if at all, Q has a plane cavity in C_k. Lemma 1 says that Q has a reflex point. □

Unfortunately, we do not know how to turn this definition into an efficient procedure to compute the reflex-free hull of a polyhedron. The process of filling in one reflex vertex can create others at reflex edges. Figure 7 illustrates one example in which filling cavities must be taken to the limit to attain the reflex-free hull.

Start with the cube $[-5,5]^3$ and subtract the following sets: $\{(x,y,z) \mid z < -3\}$, $\{(x,y,z) \mid x \in [-1,1], z > |y|\}$, $\{(x,y,z) \mid |x| \in [1,3], z > |y| - |x| + 1\}$, $\{(x,y,z) \mid x \in [3,5], 2z > y+1\}$, and $\{(x,y,z) \mid x \in [-3,-5], 2z > -y+1\}$, to obtain an object illustrated in Fig. 7. There are four labeled lines (presented in thick line segments) that are relevant in this example. We parameterize them by z. Two lines are *pivot lines*, $\alpha(z) = (-3, 1-2z, z)$ and $\delta(z) = (3, 2z-1, z)$, and two lines end at saddle points, $\beta(z) = (-1, z, z)$, and $\gamma(z) = (1, -z, z)$. With a little algebra, we observe that a plane that contains the entire line δ and point $\beta(t)$, for any chosen value of $t \geq 0$, must intersect the line $\gamma(z)$ at $z = (t+1)/6$. By symmetry, the plane containing the line α and point $\gamma(t)$ intersects the line $\beta(z)$ at $z = (t+1)/6$.

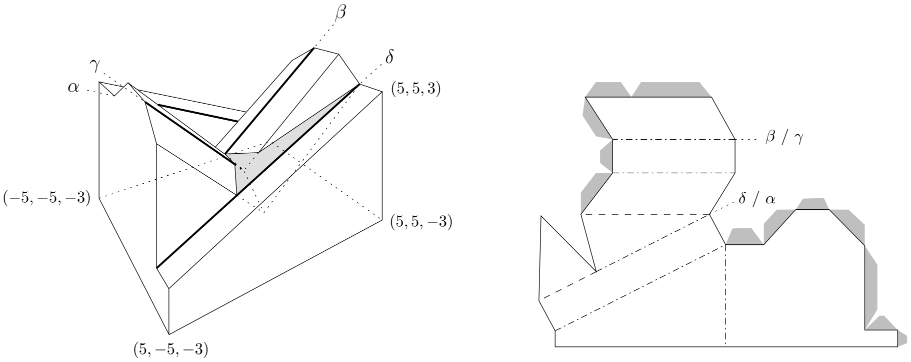


Fig. 7. A polyhedron in which each filling step creates reflex and saddle points; you can make your own by making two copies of the pattern on card stock. Make mountain (dash-dot) and valley (dash) folds shown by origami conventions, then mountain fold the dark tabs and valley fold light tabs, and glue tab faces to the underside of the model.

Initially, there are two reflex vertices with coordinates $(\pm 3, 0, -2)$. We can eliminate the first by filling the cavity defined by the plane through pivot line δ and the saddle point $\beta(0)$; this plane intersects γ at $z_1 = 1/6$. We eliminate the second by filling to the plane through α and the newly created saddle point $\gamma(z_1)$; this plane intersects β at $z_2 = 7/36$, and creates a new reflex vertex where the two filling planes meet. From now on, we fill from a pivot line to a saddle at $z_i = (z_{i-1} + 1)/6$. The reflex-free hull for this example has a reflex edge along the line through $(-1, 1/5, 1/5)$ and $(1, -1/5, 1/5)$, which happens to be the unique line incident to α, β, γ, and δ. Thus, we approach, but never reach the reflex-free hull.

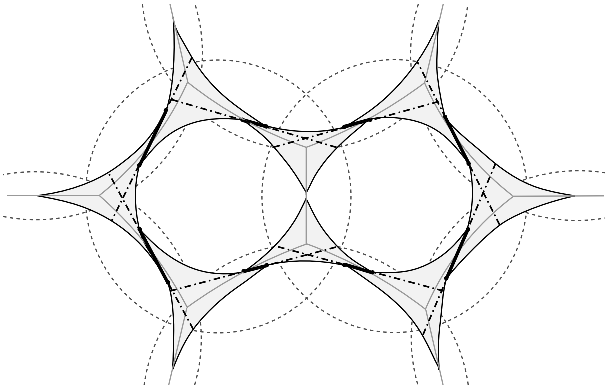


Fig. 8. Part of the reflex hull of an appropriately placed set of eight spheres. The planes determining the new boundaries of this hull are defined by a sequence of saddles in such a way that if any saddle is moved then all plane equations change.

If reflex edges incident to four polyhedron edges were the worst that could occur, we could still hope for a polynomial-time algorithm for the reflex-free hull by inspecting all 4-tuples of edges to see if they support a common line. Unfortunately, however, reflex edges may be defined by lines that hit only two polyhedron edges. Figure 8 shows such an example made of eight spheres, which could be approximated by polyhedra. In it, we have a sequence of eight geodesic triangles that share the thick segments, which are reflex edges of the hull. The extensions of reflex edges (dash-dotted) intersect within the incident geodesic triangles. The reader who would like to find an algorithm to compute reflex-free hulls is advised to build similar examples from modeling clay.

7. Other hulls

The fact that the reflex-free hull has linear complexity may not at first seem surprising. In this section, we consider some other natural definitions for hulls that have far worse complexities.

For a closed set S, we may obtain the convex hull, $CH(S)$, by removing halfspaces that do not intersect S or by taking the intersection of halfspaces that contains S. We may obtain the reflex-free hull, $Rfh(S)$, by sculpting according to Lemma 3, or by taking the intersection of reflex-free sets that contain S. In a similar manner, we can define a line hull, $LH(S)$, by removing lines that do not intersect S, or more formally by taking the intersection of sets containing S that are the complements of lines. The line hull is also known as the visual hull[10] in the context of computer vision and graphics.

Lemma 13. *For a polyhedral set S, we have $S \subseteq Rfh(S) \subseteq LH(S) \subseteq CH(S)$. In general, all inclusions are strict.*

Proof. It follows immediately from the definitions. The complement of a line is reflex-free, and a halfplane can be represented as the intersection of the complements of lines. □

In the plane, the line hull of a connected set is the same as its convex hull, but the line hull of a disconnected polyhedral set of size n may have $\Theta(n^4)$ complexity, as it is related to an arrangement of the $\Theta(n^2)$ lines tangent to pairs of vertices of S (See Fig. 9.) In $\mathbb{R}^3$, the line hull is bounded by pieces of ruled surfaces, including hyperboloids. Sergei Bespamyatnikh, in private communication, described an example of a connected polyhedral set S of size n whose line hull has $\Theta(n^9)$ complexity. Begin with the six faces of a large, axis-aligned cube, and cut a small square hole in the center of each face. Near the center of this cube we have three families of lines, each roughly parallel to one of the three axes. Block the lines parallel to the x-axis with three squares parallel to the yz plane, and cut n parallel slits in different directions each square so that the lines that do pass through these slits form $\Omega(n^3)$ hyperboloids near the center of the cube. Repeat this for the y- and z-axes, so that these hyperboloids have $\Omega(n^9)$ intersections.

Some other natural hull definitions suffer from similar complexities. One could define the *star hull*, $SH(S)$, of S as the intersection of all star-shaped polyhedra that contain S. Each point p that is not in the star hull is excluded from some star-shaped polyhedron, which says that p has a ray to infinity that does not intersect the interior of S. Thus, one

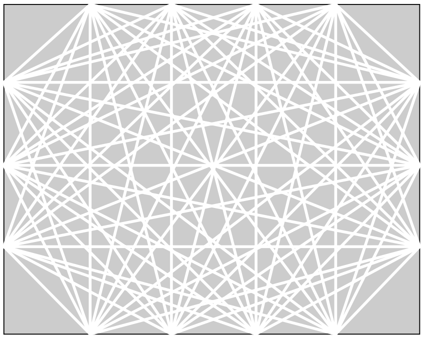


Fig. 9. Line hull of 18 black segments

could define the *ray hull* as the intersection of sets containing S that are the complements of open rays. We say that a point p of a closed set X is *externally visible* if there is a ray from p that does not intersect the interior of X. A set X is *externally visible* if every point on its boundary is externally visible. Thus, one could define the *externally visible hull* of a closed set S to be the intersection of all externally visible sets that contain S. It is not difficult to see that the star hull, ray hull, and externally visible hull are identical, and that $S \subseteq SH(S) \subseteq LH(S)$, with strict inclusion for many sets S. The reflex-free hull and star hull cannot always be ordered by inclusion: The boundary of the star hull contains any reflex vertices from S that are externally visible. The boundary of the reflex-free hull may contain points that are not externally visible, as can be seen in an example of nested tori rotated about a common axis so they form a spherical shell.

A minor modification of Bespamyatnikh's construction shows that the star hull can again be bounded by hyperboloids, and may have $\Omega(n^9)$ complexity.

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