On $\beta$-skeleton as a Subgraph of the Minimum Weight Triangulation

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Abstract

Given a set $S$ of $n$ points in the plane, a triangulation is a maximal set of non-intersecting edges connecting the points in $S$. The weight of the triangulation is the sum of the lengths of the edges. In this paper, we show that for $\beta > 1/\sin \kappa$, the $\beta$-skeleton of $S$ is a subgraph of a minimum weight triangulation of $S$, where $\kappa = \tan^{-1}(3/2\sqrt{3}) \approx \pi/3.1$. There exists a four point example such that the $\beta$-skeleton for $\beta < 1/\sin(\pi/3)$ is not a subgraph of the minimum weight triangulation.

Keywords: Minimum weight triangulation, beta skeleton, computational geometry.

1 Introduction

Let $S$ be a set of $n$ points in the plane. A triangulation $T(S)$ of $S$ is a maximal set of non-intersecting straight line edges connecting points in $S$. Let $CH(S)$ denote the set of edges bounding the convex hull of $S$. Then $|T(S)| = 3n - 3 - |CH(S)|$ [6]. The length of an edge in $T(S)$ is equal to the Euclidean distance between its two endpoints. The weight of $T(S)$ is the sum of the lengths of edges in $T(S)$. The minimum weight triangulation problem is to compute $T(S)$ with minimum weight for a given point set $S$. The problem finds applications in numerical analysis [5, 8, 18]. However, the complexity of the problem remains open.

Several heuristics have been proposed to obtain a triangulation to approximate the MWT [4, 9, 12, 13, 14]. The heuristic in [14] is known to have a bound of $O(\log n)$ on the approximation ratio in the worst case. The more recently discovered heuristic [12] computes in $O(n \log n)$ time a triangulation with constant approximation ratio. Relatively little is known about the structure of the MWT. It is shown in [7] that the shortest edge between two points in $S$ belongs to any MWT. Mark Keil [10] proves that a much larger graph, $\sqrt{2}$-skeleton, is always a subgraph of a

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MWT. The $\sqrt{2}$-skeleton is the $\beta$-skeleton defined by Kirkpatrick and Radke in [11] for $\beta = \sqrt{2}$. Given two points $x$ and $y$, define $xy$ to be the edge connecting $x$ and $y$ and define $|xy|$ to be the length of $xy$. For $\beta \geq 1$, the forbidden neighborhood of $x$ and $y$ is the union of two disks with radius $\beta |xy|/2$ that pass through both $x$ and $y$. Given a point set $S$ and $x, y \in S$, $xy$ belongs to the $\beta$-skeleton of $S$ if no point in $S$ lies in the interior of the forbidden neighborhood of $x$ and $y$. Refer to Figure 1. Let $\alpha_{xy}$ be the angle that the chord $xy$ subtends at one of the circles. Then $\beta = 1/\sin \alpha_{xy}$.

It is conjectured in [10] that the $\beta$-skeleton is a subgraph of a MWT for $\beta \geq 1/\sin (\pi/3)$. Recently, it is reported in [16] that the value of $\beta$ can be improved to $1/\sin (2\pi/7) \approx 1.279$. Yang et al. [17] formulated and proved a different property: if the union of the two disks centered at $x$ and $y$ with radius $|xy|$ is empty, then $xy$ is in a MWT (this interpretation of the original statement in [17] is from [1]). Note that the subgraph generated by the above condition and the $\beta$-skeleton do not contain each other for $\beta > 1/\sin (\pi/3)$, but for $\beta \leq 1/\sin (\pi/3)$, the $\beta$-skeleton contains the subgraph generated by the above condition.

In this paper, we show that the $\beta$-skeleton is a subgraph of a MWT, for $\beta > 1/\sin \kappa \approx 1.17682$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$. Both our result and the result in [16] are based on proving an improved version of the key lemma, Remote Length Lemma, in [10]. Moreover, the proof strategy in [10] cannot be pushed further to improve upon our result. There exists a four point example such that the $\beta$-skeleton for $\beta < 1/\sin (\pi/3) \approx 1.1547$ is not a subgraph of any MWT. Refer to Figure 1. The two circles define the forbidden region for $xy$ for $\beta_0 = 1/\sin (\pi/3)$. The triangle $axy$ is equilateral. The two shaded disks define the forbidden region for $xy$ for $\beta_1 < 1/\sin (\pi/3)$. Thus, $bx < ax = xy$. We can pick a point $c$ on the boundary of the lower shaded disk such that $bc < xy$. So $xy$ belongs to the $\beta_1$-skeleton of $\{b, c, x, y\}$ but the MWT of $\{b, c, x, y\}$ contains $bc$ instead of $xy$. After the appearance of a preliminary version of this paper [3], it has been proved recently [15] that $1/\sin \kappa$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}})$, is indeed a lower bound on $\beta$ for $\beta$-skeleton to be a subgraph of a MWT.

In Section 2, we shall review Keil’s proof. Our result is presented in Section 3.
2 Preliminaries

Keil’s proof follows the edge insertion paradigm [2]. Assume to the contrary that \( xy \) is an edge of a \( \beta \)-skeleton that does not belong to a MWT \( T \). The strategy is to add \( xy \) to \( T \) and remove the existing edges that intersect \( xy \). Then the two resulting polygonal regions on both sides of \( xy \) are retriangulated carefully to obtain a new triangulation. A contradiction is derived by arguing that the new triangulation has a smaller weight than \( T \). We describe the main ideas below. Assume throughout that \( \alpha_{xy} < \pi/3 \).

Let \( e_j, 1 \leq j \leq m \), be the edges intersected by \( xy \) and let \( |e_{j-1}| \leq |e_j|, 2 \leq j \leq m \). Let \( P \) be the polygonal region above \( xy \) to be retriangulated incrementally. (The polygonal region below \( xy \) can be dealt with similarly.) During the incremental retriangulation, we shall obtain a sequence of triangulated polygons \( P_j, 0 \leq j \leq m \), such that \( P_0 \) is the degenerate polygon \( xy \), \( P_m \) is a triangulation of \( P \), and \( P_{j-1} \subseteq P_j \). \( P_j \) is obtained from \( P_{j-1} \) by expanding \( P_{j-1} \) to include the endpoint \( v_j \) of \( e_j \) as follows (\( v_j \) is the endpoint on the same side of \( xy \) as \( P \)). If \( v_j \) lies in \( P_{j-1} \), then \( P_j = P_{j-1} \). Otherwise, \( e_j \) intersects a boundary edge \( v_k \) of \( P_{j-1} \). In general, the triangle \( v_j v_k \) contains a subsequence \( \sigma_1 \) of vertices on \( P \) from \( v_j \) to \( v_k \) and another subsequence \( \sigma_2 \) from \( v_j \) to \( v_k \). See Figure 2: the polygon with solid boundary is \( P_{j-1} \), the bold triangle is \( v_j v_k \), the polygon with dashed boundary is \( P \), the white dots inside the bold triangle is \( \sigma_1 \), and the grey dots inside the bold triangle is \( \sigma_2 \). We arbitrarily triangulate the polygon \( v_j \sigma_1 \sigma_2 v_k \) and \( P_j \) is the union of this triangulated polygon and \( P_{j-1} \). We claim that all the new edges added are shorter than \( e_j \). Thus, we shall inductively obtain a new triangulation of lesser weight than \( T \) (and so the contradiction).

The proof of the claim is as follows. All new edges added have length at most \( \max\{|v_i v_j|, |v_j v_k|, |v_i v_k|\} \). \( v_i v_k \) is shorter than \( e_{j-1} \) by induction assumption. Consider \( v_i v_j \) (\( v_j v_k \) can be handled similarly). If \( v_i \) lies in triangle \( x v_j y \), then by triangle inequality and the fact that \( \alpha_{xy} < \pi/3 \), \( v_i v_j \) is shorter than \( e_j \). Otherwise, consider the convex hull of the chain from \( x \) to \( v_j \) on \( P_j \), \( v_i \) must lie in a triangle \( v_a v_b v_j \), where \( v_a \) and \( v_b \) are hull vertices. Thus, \( |v_i v_j| \leq \max\{|v_a v_j|, |v_a v_b|, |v_b v_j|\} \). Since \( v_a \) and \( v_b \) are hull vertices, \( v_a \) and \( v_b \) were added in the growth process in the past. Thus, the edges \( v_a e_a \) and \( e_b v_b \) with endpoints \( v_a \) and \( v_b \) respectively, were processed before \( e_j \). So \( |v_a| \leq |e_j| \) and \( |v_b| \leq |e_j| \). Applying the following Lemma 1 to \( v_a v_j \) implies that \( |v_a v_j| < |e_j| \). Similarly, we obtain \( |v_b v_j| < |e_j| \) and \( |v_a v_b| < |e_j| \). Thus, \( |v_i v_j| < |e_j| \) and this completes the proof. Refer to Figure 3 for an illustration of the Remote Length Lemma.

**Lemma 1 (Remote Length Lemma)** [10] Suppose that \( \beta \geq \sqrt{2} \). Let \( x \) and \( y \) be the endpoints of an edge in the \( \beta \)-skeleton of a set \( S \) of points in the plane. Let \( p, q, r \), and \( s \) be four other distinct points of \( S \) such that \( pq \) intersects the interior of \( xy \), \( rs \) intersects the interior of \( xy \), \( pq \) and \( rs \) does not intersect the interior of each other and \( p \) and \( s \) lie on the same side of the line through \( xy \). Then either \( |qr| < |pq| \) or \( |qr| < |rs| \).

As observed in [10], for \( 1/\sin(\pi/3) \leq \beta < \sqrt{2} \), the only part of the entire proof in [10] that may fail is the Remote Length Lemma. We achieve our result by showing that the Remote Length Lemma is true for \( \beta > 1/\sin \kappa \approx 1.17682 \), where \( \kappa = \tan^{-1}(3/\sqrt{2}) \approx \pi/3.1 \).
Figure 2.

Figure 3.
3 The Proof

Let $x$ and $y$ be the endpoints of an edge in the $\beta$-skeleton of a set $S$ of points in the plane. Let $(p,q,r,s)$ be a four tuple of distinct points (not necessarily in $S$) outside or on the boundary of the forbidden neighborhood of $xy$, such that $pq$ intersects $xy$, $rs$ intersects $xy$, $pq$ and $rs$ does not intersect the interior of each other and $p$ and $s$ lie on the same side of the line through $xy$. If $|qr| \geq |pq|$ and $|qr| \geq |rs|$, then we say that $(p,q,r,s)$ satisfies the remote length exception with respect to $xy$. Refer to Figure 3. Let the two circles be $C_1$ and $C_2$. Throughout this paper, we assume that $\alpha_{xy}$ is some fixed constant such that $\alpha_{xy} < \pi/3$ and there exists some $(p,q,r,s)$ that satisfies the remote length exception with respect to $xy$.

Define $\Phi(x,y)$ be the set of four tuples of points $(p,q,r,s)$ such that $(p,q,r,s)$ satisfies the remote length exception with respect to $xy$. The basic idea of our proof is to compute the smallest value $\kappa$ for $\alpha_{xy}$ such that $\Phi(x,y) \neq \emptyset$. In other words, for all values of $\alpha_{xy} < \kappa$, $\Phi(x,y) = \emptyset$ and therefore, the Remote Length Lemma holds in general. The corresponding value, $1/\sin \kappa$, for $\beta$ will give us an improvement upon the result in [10].

Since there can be an infinite number of four tuples $(p,q,r,s)$ that belong to $\Phi(x,y)$, it is not clear how to compute $\kappa$ and hence $\beta$ directly. Instead, we restrict our attention to a critical structure that must exist in $\Phi(x,y)$ if $\Phi(x,y) \neq \emptyset$. We first fully characterize this critical structure. Select a subset $A = \{(p,q,r,s) \in \Phi(x,y) : \max(|pq|,|rs|) \text{ is minimized}\}$. Then select a subset $\Phi^*(x,y) = \{(p,q,r,s) \in A : |pq| + |rs| \text{ is minimized}\}$. $\Phi^*(x,y)$ turns out to be a singleton set containing this critical structure. Then, we compute $\kappa$ based on this knowledge. The characterization of the critical structure is given in the next section. The calculation of $\kappa$ and $\beta$ is given in Section 3.2.

3.1 Characterizing $\Phi^*(x,y)$

The main result in this section is that if $(p,q,r,s) \in \Phi^*(x,y)$, then $|qr| = |pq| = |rs|$, $\angle xy = \angle yx$ and they are obtuse. See Figure 4. There are several geometric facts Observation A, Observation B, and Observation C that we will use in our argument. Observation A refers to Figure 5(a), Observation B refers to Figure 5(b) and Observation C refers to Figure 5(c).

**Observation A** Let $cd$ be a line segment through $x$ with endpoints on $C_1$ and $C_2$. Then $|cd|$ is a continuous concave function $F$ in $\angle cxy$. Moreover, the slope of $F$ becomes zero only when $\angle cxy = \pi/2$, $F$ is symmetric around $\angle cxy = \pi/2$, and $|cd|$ is maximized when $\angle cxy = \pi/2$.

**Observation B** Let $ef$ be a line segment with endpoints $e$ on $C_1$ and $f$ on $C_2$ such that the two centers of $C_1$ and $C_2$ lie on the same side of $ef$ and $ef$ intersects the interior of $xy$. If $f$ (resp. $e$) slides on $C_2$ (resp. $C_1$) such that $ef$ rotates away from the centers and $ef$ still intersects $xy$, then $|ef|$ decreases.

**Observation C** Let $ef$ be a line segment with endpoints $e$ on $C_1$ and $f$ on $C_2$ such that the two centers of $C_1$ and $C_2$ lie on opposite sides of $ef$ and $ef$ intersects the interior of $xy$. If $f$ is closer to $y$ (resp. $x$), then sliding $f$ along $C_2$ clockwise (resp. counter-clockwise) decreases $|ef|$, provided that $ef$ still intersects $xy$. If $e$ is closer to $x$ (resp. $y$), then sliding $e$ along $C_1$
Figure 4.

Figure 5.
clockwise (resp. counter-clockwise) decreases $|ef|$, provided that $ef$ still intersects $xy$.

Lemma 2 If $(p, q, r, s) \in \Phi^*(x, y)$, then $p$ and $s$ lie on $C_1$, $p \neq s$, and $q$ and $r$ lie on $C_2$.

Proof Refer to Figure 3. If $p$ does not lie on $C_1$, then we can shorten $pq$ to make $p$ lie on $C_1$. This contradicts that $|pq| + |rs|$ is minimized. The same argument holds for $s$. So $p$ and $s$ lie on $C_1$. Assume to the contrary that $p = s$. Then $qr$ is the longest side of the triangle $pqr$, which implies that $\angle qpr \geq \pi/3$. However, $\angle xyz \geq \angle qpr \geq \pi/3$ which contradicts our assumption that $\alpha_{xy} = \angle xyz < \pi/3$. In the following, assume to the contrary that $q$ does not lie on $C_2$. The treatment for $r$ is similar.

Case(1) : $\angle qpr \geq \pi/2$. Refer to Figure 6(a). Let $C'$ be the circle with center $p$ and radius $|pq|$. Draw a circular arc $A$ through $q$ with center $r$ and radius $|qr|$ such that $A$ does not intersect $C_2$ or $rs$ and $A$ intersects $C'$ exactly once at $q$. The endpoint $q'$ of $A$ shown in the figure must lie inside $C'$ but outside $C_2$. Thus $|q'r| = |qr|$, $\max(|pq'|, |rs|) = \max(|pq|, |rs|)$, but $|pq'| < |pq|$. Hence, $(p, q', r, s) \in \Phi(x, y)$ and $|pq'| + |rs| < |pq| + |rs|$. This contradicts our assumption that $|pq| + |rs|$ is the minimum possible.

Case(2) : $\angle qpr < \pi/2$. Refer to Figure 6(b). Let $C'$ be the circle with center $p$ and radius $|pq|$. Draw a circular arc $A$ through $q$ with center $r$ and radius $|qr|$ such that $A$ does not intersect $C_2$ or $rs$ and $A$ intersects $C'$ exactly once at $q$. The endpoint $q_0$ of $A$ shown in the figure must lie outside the quadrilateral $pqrs$ and $C_2$ but inside $C'$. If $pq$ does not pass through $x$, then $A$ can be made short enough such that $pq_0$ intersects $xy$. Then $(p, q_0, r, s) \in \Phi(x, y)$ and $|pq_0| < |pq|$ which contradicts the minimality of $|pq| + |rs|$. Suppose that $pq$ passes through $x$. Draw a line segment from $q_0$ through $x$ to $p_0$ on $C_1$. Let the other endpoint of $A$ be $q_1$. Draw another line segment from $q_1$ through $x$ to $p_1$ on $C_1$. Denote by $B$ the circular arc on $C_1$ traversed clockwise from $p_0$ to $p_1$. For an arbitrary point $q_t$ on $A$, define $p_t$ to be the point on $B$ such that $p_tq_t$ passes through $x$. See Figures 7. Let $\theta_0 = \angle q_0xy$, $\theta_1 = \angle q_1xy$, and $c = \angle rxy$. Let $\theta^* = \angle qx_0$ and $\theta \neq \angle qx_0$. Then

\[
|q_0x| = |rx| \cos(\theta - c) + \sqrt{|q_0r|^2 - |rx|^2 \sin^2(\theta - c)} \quad |p_0x| = |xy| \sin(\theta - \alpha_{xy})/\sin \alpha_{xy}.
\]
It is clear from the figure that both \(|q_0x|\) and \(|p_1x|\) are concave in \([\theta_1, \theta_0]\). Moreover, since \(|q_0x|\) and \(|p_1x|\) are trigonometric, they are concave functions with a unique maximum in \([\theta_1, \theta_0]\). Therefore, within \([\theta_1, \theta_0]\), \(|p_1q_i| = |p_1x| + |q_0x|\) must have at most one stationary point (the unique maximum if it exists) and \(|p_1q_i|\) achieves the minimum at \(\theta_0\) or \(\theta_1\) or both. Since \(\theta^* \in (\theta_1, \theta_0)\), we conclude that \(|p_0q_0| < |pq|\) or \(|p_1q_1| < |pq|\). So \((p_0, q_0, r, s) \in \Phi(x, y)\) or \((p_1, q_1, r, s) \in \Phi(x, y)\) and this contradicts the minimality of \(|pq| + |rs|\). □

**Lemma 3** Let \(v\) and \(w\) be the centers of \(C_1\) and \(C_2\). If \((p, q, r, s) \in \Phi^*(x, y)\), then \(v\) and \(w\) lie on the right of \(pq\) and on the left of \(rs\), respectively.

**Proof** By Lemma 2, \(p\) and \(s\) lie on \(C_1\) and \(q\) and \(r\) lie on \(C_2\). Assume to the contrary that the lemma is not true. Then either \(v\) and \(w\) lie on the same side of \(pq\) and \(rs\) (Case(1)), or \(v\) and \(w\) lie on opposite sides of \(pq\) or \(rs\) (Case(2)).
Case(1): Assume without loss of generality that \( v \) and \( w \) lie on the left of \( pq \) and \( rs \). Refer to Figure 8(a). Since \( rs \) lies to the right of \( pq \), \( pq \) does not pass through \( y \). Since \( p \neq s \), by Observation B, we can slide \( p \) along \( C_1 \) counter-clockwisely to decrease \( |pq| \), but this contradicts the minimality of \( |pq| + |rs| \).

Case(2): Assume without loss of generality that \( v \) and \( w \) lie on opposite sides of \( pq \). Refer to Figure 8(b). By Observation C, we can slide \( p \) along \( C_1 \) either clockwise or counter-clockwisely to decrease \( |pq| \), depending on whether \( p \) is closer to \( x \) or \( y \). This contradicts the minimality of \( |pq| + |rs| \). \( \square \)

**Lemma 4** If \((p, q, r, s) \in \Phi^*(x, y)\), then \( pq \) passes through \( x \) and \( rs \) passes through \( y \).

**Proof** First, \((p, q, r, s) \) satisfies Lemma 2 and Lemma 3. If \( pq \) (resp. \( rs \)) does not pass through \( x \) (resp. \( y \)), then by Observation B, we can slide \( p \) along \( C_1 \) clockwise (resp. \( s \) along \( C_1 \) counter-clockwisely) and decrease \( |pq| \) (resp. \( |rs| \)). This contradicts the minimality of \( |pq| + |rs| \). \( \square \)

**Lemma 5** If \((p, q, r, s) \in \Phi^*(x, y)\), then \(|qr| = |pq| = |rs|\) and \( \angle qxy \) and \( \angle rxy \) are obtuse.

**Proof** First, \((p, q, r, s) \) satisfies Lemmas 2–4. Since \((p, q, r, s) \in \Phi^*(x, y)\), \(|qr| \geq \max(|pq|, |rs|)\). Without loss of generality, assume that \(|pq| = \max(|pq|, |rs|)\). Let \( w \) be the center of \( C_2 \). For brevity, rotating \( pq \) about \( x \) or \( rs \) about \( y \) means that we keep \( p \) and \( s \) on \( C_1 \) and \( q \) and \( r \) on \( C_2 \) during the rotation.

Assume to the contrary that \(|qr| > |pq|\). If \( \angle pxy \leq \pi/2 \), then we can rotate \( pq \) about \( x \) counter-clockwisely by an infinitesimal amount and still maintain that \(|qr| > \max(|pq|, |rs|)\). However, by Observation A, \(|pq| \) decreases which contradicts the minimality of \(|pq| + |rs|\). If \( \angle pxy > \pi/2 \), then \( \angle qxy < \pi/2 \). We can rotate \( pq \) about \( x \) clockwise by an infinitesimal amount and still maintain that \(|qr| > \max(|pq|, |rs|)\). By Observation A, \(|pq| \) decreases which contradicts the minimality of \(|pq| + |rs|\). Hence, we conclude that \(|qr| = |pq|\).

We claim that \( w \) does not lie inside the quadrilateral \( pqrw \) or on \( pq \) or on \( rs \). Assume to the contrary this is not true. Observe that \( \angle rwy < \pi/2 \); otherwise, we can rotate \( rs \) about \( y \) clockwise to increase \(|qr| \) and decrease \(|rs| \), which contradicts the minimality of \(|pq| + |rs|\). By a similar argument, \( \angle qxy \) must also be acute. If \(|qr| = |pq| > |rs|\), then we can rotate \( rs \) about \( y \) clockwise by an infinitesimal amount to increase \(|qr| \) and \(|rs| \) (\(|pq| \) remains unchanged) such that \(|qr| > |pq| \) > \(|rs|\). But then we can rotate \( pq \) about \( x \) clockwise by an infinitesimal amount to decrease \(|qr| \) and \(|pq| \) such that \(|qr| > |pq| \) > \(|rs|\). However, we have decreased \( \max(|pq|, |rs|) \) which contradicts its minimality by assumption. Therefore, \(|qr| = |pq| = |rs|\). By Observation A, \( pqrw \) must be a regular trapezoid with \(|ps| > |qr| = |pq| = |rs|\). See Figure 9. Now, we can rotate \( pq \) about \( x \) clockwise and \( rs \) about \( y \) counter-clockwisely by some amount to decrease \(|pq| \) and \(|rs| \), while maintaining that \(|ps| > \max(|pq|, |rs|)\). Then we can switch the roles of \( qr \) and \( ps \) to obtain the four tuple \((r, s, p, q) \in \Phi(x, y)\) with a smaller \( \max(|pq|, |rs|) \). This contradicts our assumption. In all, we conclude that \( w \) does not lie inside \( pqrw \) or on \( pq \) or on \( rs \). So \( w \) either lies outside \( pqrw \) or on \( qr \).

Suppose that \( w \) lies on \( qr \). Then \( qr \) must be horizontal in order that \( \max(|pq|, |rs|) \) is
minimized. At this position, $|pq| = |rs|$. Since we have proved before that $|qr| = |pq|$, we conclude that $|qr| = |pq| = |rs|$. It is clear that both $\angle qxy$ and $\angle ryx$ are obtuse at this position.

Suppose that $w$ lies outside $pqrs$. Assume to the contrary that $|qr| > |rs|$. Observe that $\angle ryx$ and $\angle qxy$ are obtuse; otherwise, we can rotate $rs$ about $y$ counter-clockwise (resp. rotate $pq$ about $x$ clockwise) to decrease $|rs|$ (resp. decrease $|pq|$) and increase $|qr|$. This contradicts the minimality of $|pq| + |rs|$. We rotate $rs$ about $y$ counter-clockwise by an infinitesimal amount to decrease $|qr|$ and $|rs|$ ($|pq|$ remains unchanged) such that $|qr| > |pq| > |rs|$. Now, we can rotate $pq$ about $x$ counter-clockwise by an infinitesimal amount to decrease $|qr|$ and $|pq|$ such that $|qr| > |pq| > |rs|$. But we have decreased $\max(|pq|, |rs|)$ and this contradicts our assumption. Hence, $|qr| = |pq| = |rs|$ and this completes the proof. $\square$

By Observation A and Lemma 5, we conclude that every element $(p, q, r, s)$ in $\Phi^*(x, y)$ represents a regular trapezoid as shown in Figure 10.
3.2 Calculating $\beta$

Consider a $(p, q, r, s) \in \Phi^*(x, y)$. Let $\angle pqr = \theta$. By applying the sine law to triangle $qrx$ and $rsx$, we obtain the equalities $|rx|/\sin \theta = |qr|/\sin(2\theta - \alpha_{xy})$ and $|rx|/\sin \alpha_{xy} = |rs|/\sin 2\alpha_{xy}$. By eliminating $|rx|$ from the above equations and canceling $|qr|$ and $|rs|$, we obtain $2\sin \theta \cos \alpha_{xy} = \sin(2\theta - \alpha_{xy})$. By rearranging terms, we get

$$\tan \alpha_{xy} = \frac{2\sin \theta \cos \theta - 1}{\cos 2\theta}.$$  \hspace{1cm} (1)

For a fixed $\alpha_{xy}$, we can solve Equation 1 for the the smallest positive $\theta$. This corresponds to minimizing $\max(|pq|, |rs|)$ and minimizing $|pq| + |rs|$. Thus, $\Phi^*(x, y)$ is a singleton set.

Our goal is to find the smallest $\alpha_{xy}$ such that $\Phi(x, y) \neq \emptyset$. Therefore, we differentiate Equation 1 with respect to $\theta$ and set $d(\alpha_{xy})/d\theta = 0$ to obtain $\cos \theta \cos \alpha_{xy} = \cos(2\theta - \alpha_{xy})$. By rearranging terms, we get

$$\tan \alpha_{xy} = \frac{\cos \theta - \cos 2\theta}{\sin 2\theta}.$$ \hspace{1cm} (2)

By equating Equations 1 and 2, we obtain

$$2\sin \theta \cos \theta - 1 \sin 2\theta = (\cos \theta - \cos 2\theta) \cos 2\theta \Rightarrow 4(1 - \cos^2 \theta) \cos \theta = (2 \cos^2 \theta - 1) \cos \theta - 2 \cos 2 \theta + 1 \Rightarrow 2 \cos^2 \theta + 2 \cos \theta - 1 = 0 \Rightarrow \cos \theta = \frac{\sqrt{3} - 1}{2} \text{ as } \cos \theta > 0.$$

Substituting $\cos \theta = (\sqrt{3} - 1)/2$ into Equation 1, we obtain $\alpha_{xy} = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$. The corresponding $\beta$ value is slightly less than 1.17682. Thus, we conclude that for any $\alpha_{xy} \geq \tan^{-1}(3/\sqrt{2\sqrt{3}})$, $\Phi(x, y) \neq \emptyset$. Conversely, the Remote Length Lemma is true for any $\alpha_{xy} < \tan^{-1}(3/\sqrt{2\sqrt{3}})$. This completes the proof of our main result.

**Theorem 1** Given a set $S$ of points in the plane, the $\beta$-skeleton of $S$ is a subgraph of a minimum weight triangulation of $S$ for any $\beta > 1/\sin(\tan^{-1}(3/\sqrt{2\sqrt{3}}))$.

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**References**


