

On β -skeleton as a Subgraph of the Minimum Weight Triangulation

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Abstract

Given a set S of n points in the plane, a triangulation is a maximal set of non-intersecting edges connecting the points in S . The weight of the triangulation is the sum of the lengths of the edges. In this paper, we show that for $\beta > 1/\sin \kappa$, the β -skeleton of S is a subgraph of a minimum weight triangulation of S , where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$. There exists a four point example such that the β -skeleton for $\beta < 1/\sin(\pi/3)$ is not a subgraph of the minimum weight triangulation.

Keywords: Minimum weight triangulation, beta skeleton, computational geometry.

1 Introduction

Let S be a set of n points in the plane. A *triangulation* $T(S)$ of S is a maximal set of non-intersecting straight line edges connecting points in S . Let $CH(S)$ denote the set of edges bounding the convex hull of S . Then $|T(S)| = 3n - 3 - |CH(S)|$ [6]. The *length* of an edge in $T(S)$ is equal to the Euclidean distance between its two endpoints. The *weight* of $T(S)$ is the sum of the lengths of edges in $T(S)$. The *minimum weight triangulation problem* is to compute $T(S)$ with minimum weight for a given point set S . The problem finds applications in numerical analysis [5, 8, 18]. However, the complexity of the problem remains open.

Several heuristics have been proposed to obtain a triangulation to approximate the MWT [4, 9, 12, 13, 14]. The heuristic in [14] is known to have a bound of $O(\log n)$ on the approximation ratio in the worst case. The more recently discovered heuristic [12] computes in $O(n \log n)$ time a triangulation with constant approximation ratio. Relatively little is known about the structure of the MWT. It is shown in [7] that the shortest edge between two points in S belongs to any MWT. Mark Keil [10] proves that a much larger graph, $\sqrt{2}$ -skeleton, is always a subgraph of a

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MWT. The $\sqrt{2}$ -skeleton is the β -skeleton defined by Kirkpatrick and Radke in [11] for $\beta = \sqrt{2}$. Given two points x and y , define xy to be the edge connecting x and y and define $|xy|$ to be the length of xy . For $\beta \geq 1$, the *forbidden neighborhood* of x and y is the union of two disks with radius $\beta|xy|/2$ that pass through both x and y . Given a point set S and $x, y \in S$, xy belongs to the β -skeleton of S if no point in S lies in the interior of the forbidden neighborhood of x and y . Refer to Figure 1. Let α_{xy} be the angle that the chord xy subtends at one of the circles. Then $\beta = 1/\sin \alpha_{xy}$.

It is conjectured in [10] that the β -skeleton is a subgraph of a MWT for $\beta \geq 1/\sin(\pi/3)$. Recently, it is reported in [16] that the value of β can be improved to $1/\sin(2\pi/7) \approx 1.279$. Yang *et al.* [17] formulated and proved a different property: if the union of the two disks centered at x and y with radius $|xy|$ is empty, then xy is in a MWT (this interpretation of the original statement in [17] is from [1]). Note that the subgraph generated by the above condition and the β -skeleton do not contain each other for $\beta > 1/\sin(\pi/3)$, but for $\beta \leq 1/\sin(\pi/3)$, the β -skeleton contains the subgraph generated by the above condition.

In this paper, we show that the β -skeleton is a subgraph of a MWT, for $\beta > 1/\sin \kappa \approx 1.17682$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$. Both our result and the result in [16] are based on proving an improved version of the key lemma, Remote Length Lemma, in [10]. Moreover, the proof strategy in [10] cannot be pushed further to improve upon our result. There exists a four point example such that the β -skeleton for $\beta < 1/\sin(\pi/3) \approx 1.1547$ is not a subgraph of any MWT. Refer to Figure 1. The two circles define the forbidden region for xy for $\beta_0 = 1/\sin(\pi/3)$. The triangle axy is equilateral. The two shaded disks define the forbidden region for xy for $\beta_1 < 1/\sin(\pi/3)$. Thus, $bx < ax = xy$. We can pick a point c on the boundary of the lower shaded disk such that $bc < xy$. So xy belongs to the β_1 -skeleton of $\{b, c, x, y\}$ but the MWT of $\{b, c, x, y\}$ contains bc instead of xy . After the appearance of a preliminary version of this paper [3], it has been proved recently [15] that $1/\sin \kappa$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}})$, is indeed a lower bound on β for β -skeleton to be a subgraph of a MWT.

In Section 2, we shall review Keil's proof. Our result is presented in Section 3.

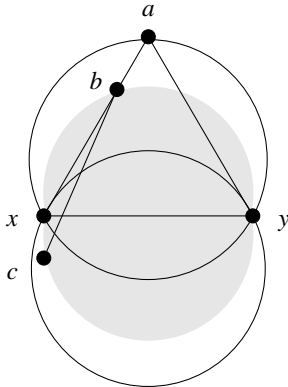


Figure 1.

2 Preliminaries

Keil's proof follows the edge insertion paradigm [2]. Assume to the contrary that xy is an edge of a β -skeleton that does not belong to a MWT \mathcal{T} . The strategy is to add xy to \mathcal{T} and remove the existing edges that intersect xy . Then the two resulting polygonal regions on both sides of xy are retriangulated carefully to obtain a new triangulation. A contradiction is derived by arguing that the new triangulation has a smaller weight than \mathcal{T} . We describe the main ideas below. Assume throughout that $\alpha_{xy} < \pi/3$.

Let e_j , $1 \leq j \leq m$, be the edges intersected by xy and let $|e_{j-1}| \leq |e_j|$, $2 \leq j \leq m$. Let P be the polygonal region above xy to be retriangulated incrementally. (The polygonal region below xy can be dealt with similarly.) During the incremental retriangulation, we shall obtain a sequence of triangulated polygons P_j , $0 \leq j \leq m$, such that P_0 is the degenerate polygon xy , P_m is a triangulation of P , and $P_{j-1} \subseteq P_j$. P_j is obtained from P_{j-1} by expanding P_{j-1} to include the endpoint v_j of e_j as follows (v_j is the endpoint on the same side of xy as P). If v_j lies in P_{j-1} , then $P_j = P_{j-1}$. Otherwise, e_j intersects a boundary edge $v_i v_k$ of P_{j-1} . In general, the triangle $v_i v_j v_k$ contains a subsequence σ_1 of vertices on P from v_i to v_j and another subsequence σ_2 from v_j to v_k . See Figure 2: the polygon with solid boundary is P_{j-1} , the bold triangle is $v_i v_j v_k$, the polygon with dashed boundary is P , the white dots inside the bold triangle is σ_1 , and the grey dots inside the bold triangle is σ_2 . We arbitrarily triangulate the polygon $v_i \sigma_1 v_j \sigma_2 v_k$ and P_j is the union of this triangulated polygon and P_{j-1} . We claim that all the new edges added are shorter than e_j . Thus, we shall inductively obtain a new triangulation of lesser weight than \mathcal{T} (and so the contradiction).

The proof of the claim is as follows. All new edges added have length at most $\max\{|v_i v_j|, |v_j v_k|, |v_i v_k|\}$. $v_i v_k$ is shorter than e_{j-1} by induction assumption. Consider $v_i v_j$ ($v_j v_k$ can be handled similarly). If v_i lies in triangle $xv_j y$, then by triangle inequality and the fact that $\alpha_{xy} < \pi/3$, $v_i v_j$ is shorter than e_j . Otherwise, consider the convex hull of the chain from x to v_j on P_j . v_i must lie in a triangle $v_a v_b v_j$, where v_a and v_b are hull vertices. Thus, $|v_i v_j| \leq \max\{|v_a v_j|, |v_a v_b|, |v_b v_j|\}$. Since v_a and v_b are hull vertices, v_a and v_b were added in the growth process in the past. Thus, the edges e_a and e_b , with endpoints v_a and v_b respectively, were processed before e_j . So $|e_a| \leq |e_j|$ and $|e_b| \leq |e_j|$. Applying the following Lemma 1 to $v_a v_j$ implies that $|v_a v_j| < |e_j|$. Similarly, we obtain $|v_b v_j| < |e_j|$ and $|v_a v_b| < |e_j|$. Thus, $|v_i v_j| < |e_j|$ and this completes the proof. Refer to Figure 3 for an illustration of the Remote Length Lemma.

Lemma 1 (Remote Length Lemma) [10] *Suppose that $\beta \geq \sqrt{2}$. Let x and y be the endpoints of an edge in the β -skeleton of a set S of points in the plane. Let p , q , r , and s be four other distinct points of S such that pq intersects the interior of xy , rs intersects the interior of xy , pq and rs does not intersect the interior of each other and p and s lie on the same side of the line through xy . Then either $|qr| < |pq|$ or $|qr| < |rs|$.*

As observed in [10], for $1/\sin(\pi/3) \leq \beta < \sqrt{2}$, the only part of the entire proof in [10] that may fail is the Remote Length Lemma. We achieve our result by showing that the Remote Length Lemma is true for $\beta > 1/\sin \kappa \approx 1.17682$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$.

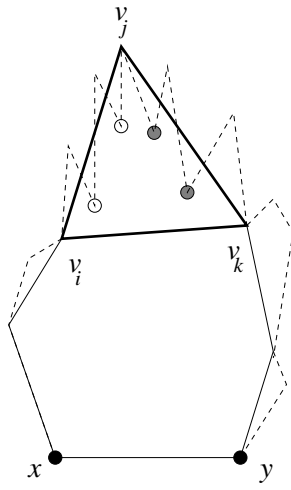


Figure 2.

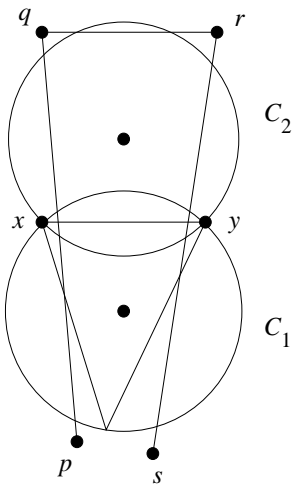


Figure 3.

3 The proof

Let x and y be the endpoints of an edge in the β -skeleton of a set S of points in the plane. Let (p, q, r, s) be a four tuple of distinct points (not necessarily in S) outside or on the boundary of the forbidden neighborhood of xy , such that pq intersects xy , rs intersects xy , pq and rs does not intersect the interior of each other and p and s lie on the same side of the line through xy . If $|qr| \geq |pq|$ and $|qr| \geq |rs|$, then we say that (p, q, r, s) satisfies the *remote length exception* with respect to xy . Refer to Figure 3. Let the two circles be C_1 and C_2 . Throughout this paper, we assume that α_{xy} is some fixed constant such that $\alpha_{xy} < \pi/3$ and there exists some (p, q, r, s) that satisfies the remote length exception with respect to xy .

Define $\Phi(x, y)$ be the set of four tuples of points (p, q, r, s) such that (p, q, r, s) satisfies the remote length exception with respect to xy . The basic idea of our proof is to compute the smallest value κ for α_{xy} such that $\Phi(x, y) \neq \emptyset$. In other words, for all values of $\alpha_{xy} < \kappa$, $\Phi(x, y) = \emptyset$ and therefore, the Remote Length Lemma holds in general. The corresponding value, $1/\sin \kappa$, for β will give us an improvement upon the result in [10].

Since there can be an infinite number of four tuples (p, q, r, s) that belong to $\Phi(x, y)$, it is not clear how to compute κ and hence β directly. Instead, we restrict our attention to a critical structure that must exist in $\Phi(x, y)$ if $\Phi(x, y) \neq \emptyset$. We first fully characterize this critical structure. Select a subset $\mathcal{A} = \{(p, q, r, s) \in \Phi(x, y) : \max(|pq|, |rs|) \text{ is minimized}\}$. Then select a subset $\Phi^*(x, y) = \{(p, q, r, s) \in \mathcal{A} : |pq| + |rs| \text{ is minimized}\}$. $\Phi^*(x, y)$ turns out to be a singleton set containing this critical structure. Then, we compute κ based on this knowledge. The characterization of the critical structure is given in the next section. The calculation of κ and β is given in Section 3.2.

3.1 Characterizing $\Phi^*(x, y)$

The main result in this section is that if $(p, q, r, s) \in \Phi^*(x, y)$, then $|qr| = |pq| = |rs|$, $\angle qxy = \angle ryx$ and they are obtuse. See Figure 4. There are several geometric facts Observation A, Observation B, and Observation C that we will use in our argument. Observation A refers to Figure 5(a), Observation B refers to Figure 5(b) and Observation C refers to Figure 5(c).

Observation A *Let cd be a line segment through x with endpoints on C_1 and C_2 . Then $|cd|$ is a continuous concave function F in $\angle cxy$. Moreover, the slope of F becomes zero only when $\angle cxy = \pi/2$, F is symmetric around $\angle cxy = \pi/2$, and $|cd|$ is maximized when $\angle cxy = \pi/2$.*

Observation B *Let ef be a line segment with endpoints e on C_1 and f on C_2 such that the two centers of C_1 and C_2 lie on the same side of ef and ef intersects the interior of xy . If f (resp. e) slides on C_2 (resp. C_1) such that ef rotates away from the centers and ef still intersects xy , then $|ef|$ decreases.*

Observation C *Let ef be a line segment with endpoints e on C_1 and f on C_2 such that the two centers of C_1 and C_2 lie on opposite sides of ef and ef intersects the interior of xy . If f is closer to y (resp. x), then sliding f along C_2 clockwise (resp. counter-clockwise) decreases $|ef|$, provided that ef still intersects xy . If e is closer to x (resp. y), then sliding e along C_1*

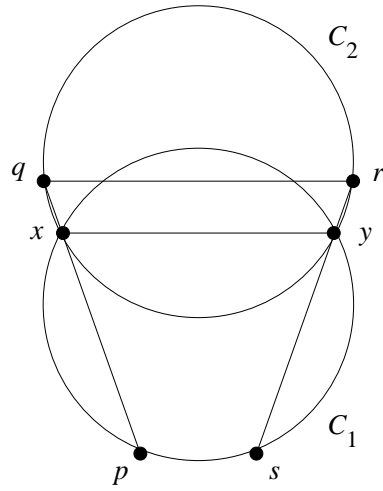
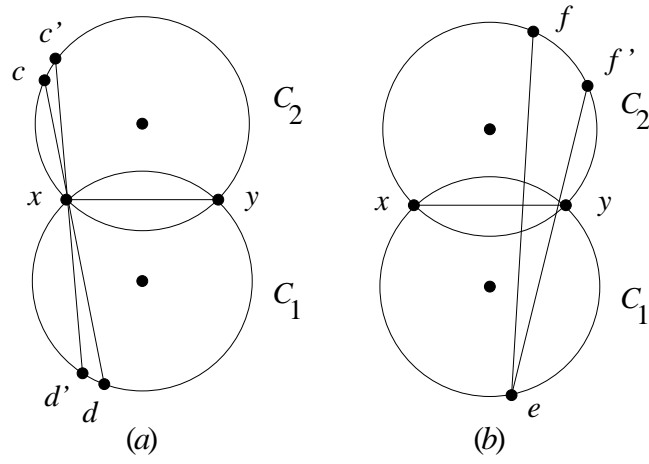
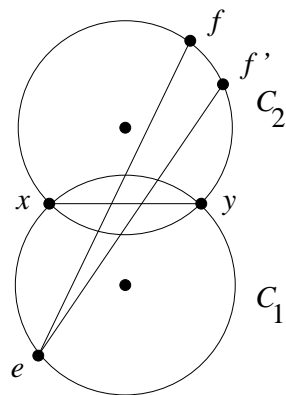


Figure 4.



(a)

(b)



(c)

Figure 5.

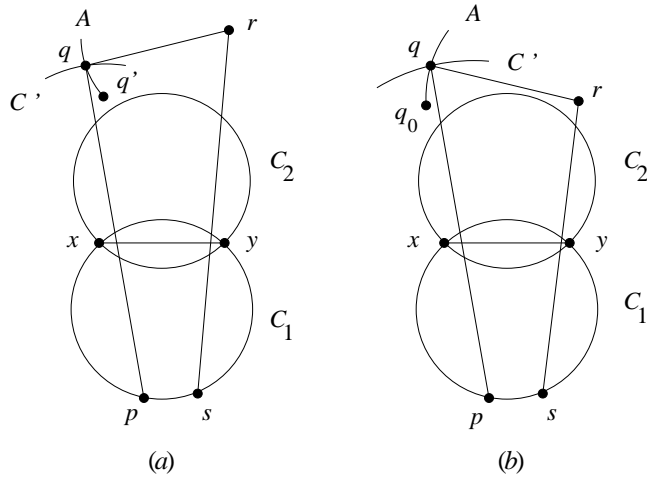


Figure 6.

clockwisely (resp. counter-clockwisely) decreases $|ef|$, provided that ef still intersects xy .

Lemma 2 *If $(p, q, r, s) \in \Phi^*(x, y)$, then p and s lie on C_1 , $p \neq s$, and q and r lie on C_2 .*

Proof Refer to Figure 3. If p does not lie on C_1 , then we can shorten pq to make p lie on C_1 . This contradicts that $|pq| + |rs|$ is minimized. The same argument holds for s . So p and s lie on C_1 . Assume to the contrary that $p = s$. Then qr is the longest side of the triangle pqr , which implies that $\angle qpr \geq \pi/3$. However, $\angle xpy \geq \angle qpr \geq \pi/3$ which contradicts our assumption that $\alpha_{xy} = \angle xpy < \pi/3$. In the following, assume to the contrary that q does not lie on C_2 . The treatment for r is similar.

Case(1) : $\angle rqp \geq \pi/2$. Refer to Figure 6(a). Let C' be the circle with center p and radius $|pq|$. Draw a circular arc A through q with center r and radius $|qr|$ such that A does not intersect C_2 or rs and A intersects C' exactly once at q . The endpoint q' of A shown in the figure must lie inside C' but outside C_2 . Thus $|q'r| = |qr|$, $\max(|pq'|, |rs|) \leq \max(|pq|, |rs|)$, but $|pq'| < |pq|$. Hence, $(p, q', r, s) \in \Phi(x, y)$ and $|pq'| + |rs| < |pq| + |rs|$. This contradicts our assumption that $|pq| + |rs|$ is the minimum possible.

Case(2) : $\angle rqp < \pi/2$. Refer to Figure 6(b). Let C' be the circle with center p and radius $|pq|$. Draw a circular arc A through q with center r and radius $|qr|$ such that A does not intersect C_2 or rs and A intersects C' exactly once at q . The endpoint q_0 of A shown in the figure must lie outside the quadrilateral $pqrs$ and C_2 but inside C' . If pq does not pass through x , then A can be made short enough such that pq_0 intersects xy . Then $(p, q_0, r, s) \in \Phi(x, y)$ and $|pq_0| < |pq|$ which contradicts the minimality of $|pq| + |rs|$. Suppose that pq passes through x . Draw a line segment from q_0 through x to p_0 on C_1 . Let the other endpoint of A be q_1 . Draw another line segment from q_1 through x to p_1 on C_1 . Denote by B the circular arc on C_1 traversed clockwise from p_0 to p_1 . For an arbitrary point q_t on A , define p_t to be the point on B such that p_tq_t passes through x . See Figures 7. Let $\theta_0 = \angle q_0xy$, $\theta_1 = \angle q_1xy$, and $c = \angle rxy$. Let $\theta^* = \angle qxy$ and $\theta = \angle q_txy$. Then

$$\begin{aligned} |q_t x| &= |rx| \cos(\theta - c) + \sqrt{|q_t r|^2 - |rx|^2 \sin^2(\theta - c)} \\ |p_t x| &= |xy| \sin(\theta - \alpha_{xy}) / \sin \alpha_{xy}. \end{aligned}$$

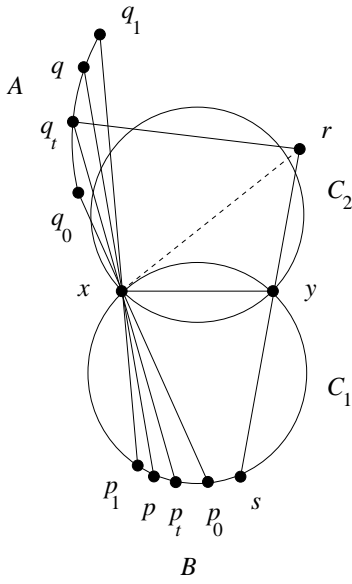


Figure 7.

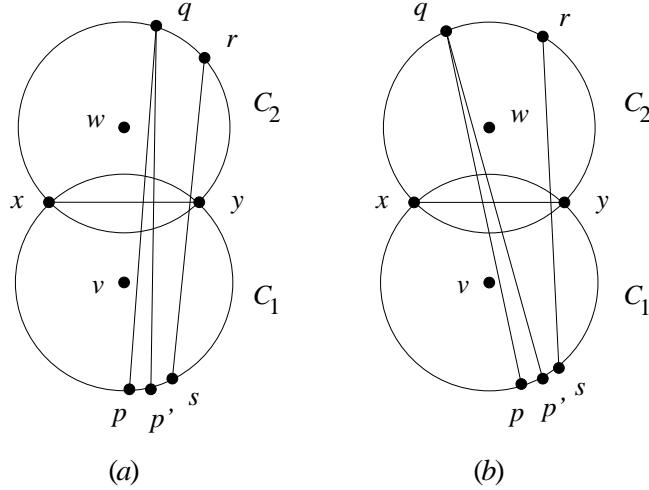


Figure 8.

It is clear from the figure that both $|q_t x|$ and $|p_t x|$ are concave in $[\theta_1, \theta_0]$. Moreover, since $|q_t x|$ and $|p_t x|$ are trigonometric, they are concave functions with a unique maximum in $[\theta_1, \theta_0]$. Therefore, within $[\theta_1, \theta_0]$, $|p_t q_t| = |p_t x| + |q_t x|$ must have at most one stationary point (the unique maximum if it exists) and $|p_t q_t|$ achieves the minimum at θ_0 or θ_1 or both. Since $\theta^* \in (\theta_1, \theta_0)$, we conclude that $|p_0 q_0| < |pq|$ or $|p_1 q_1| < |pq|$. So $(p_0, q_0, r, s) \in \Phi(x, y)$ or $(p_1, q_1, r, s) \in \Phi(x, y)$ and this contradicts the minimality of $|pq| + |rs|$. \square

Lemma 3 *Let v and w be the centers of C_1 and C_2 . If $(p, q, r, s) \in \Phi^*(x, y)$, then v and w lie on the right of pq and on the left of rs , respectively.*

Proof By Lemma 2, p and s lie on C_1 and q and r lie on C_2 . Assume to the contrary that the lemma is not true. Then either v and w lie on the same side of pq and rs (Case(1)), or v and w lie on opposite sides of pq or rs (Case(2)).

Case(1) : Assume without loss of generality that v and w lie on the left of pq and rs . Refer to Figure 8(a). Since rs lies to the right of pq , pq does not pass through y . Since $p \neq s$, by Observation B, we can slide p along C_1 counter-clockwisely to decrease $|pq|$, but this contradicts the minimality of $|pq| + |rs|$.

Case(2) : Assume without loss of generality that v and w lie on opposite sides of pq . Refer to Figure 8(b). By Observation C, we can slide p along C_1 either clockwise or counter-clockwisely to decrease $|pq|$, depending on whether p is closer to x or y . This contradicts the minimality of $|pq| + |rs|$. \square

Lemma 4 *If $(p, q, r, s) \in \Phi^*(x, y)$, then pq passes through x and rs passes through y .*

Proof First, (p, q, r, s) satisfies Lemma 2 and Lemma 3. If pq (resp. rs) does not pass through x (resp. y), then by Observation B, we can slide p along C_1 clockwise (resp. s along C_1 counter-clockwisely) and decrease $|pq|$ (resp. $|rs|$). This contradicts the minimality of $|pq| + |rs|$. \square

Lemma 5 *If $(p, q, r, s) \in \Phi^*(x, y)$, then $|qr| = |pq| = |rs|$ and $\angle qxy$ and $\angle ryx$ are obtuse.*

Proof First, (p, q, r, s) satisfies Lemmas 2–4. Since $(p, q, r, s) \in \Phi^*(x, y)$, $|qr| \geq \max(|pq|, |rs|)$. Without loss of generality, assume that $|pq| = \max(|pq|, |rs|)$. Let w be the center of C_2 . For brevity, *rotating pq about x or rs about y* means that we keep p and s on C_1 and q and r on C_2 during the rotation.

Assume to the contrary that $|qr| > |pq|$. If $\angle pxy \leq \pi/2$, then we can rotate pq about x counter-clockwisely by an infinitesimal amount and still maintain that $|qr| > \max(|pq|, |rs|)$. However, by Observation A, $|pq|$ decreases which contradicts the minimality of $|pq| + |rs|$. If $\angle pxy > \pi/2$, then $\angle qxy < \pi/2$. We can rotate pq about x clockwise by an infinitesimal amount and still maintain that $|qr| > \max(|pq|, |rs|)$. By Observation A, $|pq|$ decreases which contradicts the minimality of $|pq| + |rs|$. Hence, we conclude that $|qr| = |pq|$.

We claim that w does not lie inside the quadrilateral $pqrs$ or on pq or on rs . Assume to the contrary this is not true. Observe that $\angle ryx < \pi/2$; otherwise, we can rotate rs about y clockwise to increase $|qr|$ and decrease $|rs|$, which contradicts the minimality of $|pq| + |rs|$. By a similar argument, $\angle qxy$ must also be acute. If $|qr| = |pq| > |rs|$, then we can rotate rs about y clockwise by an infinitesimal amount to increase $|qr|$ and $|rs|$ ($|pq|$ remains unchanged) such that $|qr| > |pq| > |rs|$. But then we can rotate pq about x clockwise by an infinitesimal amount to decrease $|qr|$ and $|pq|$ such that $|qr| > |pq| > |rs|$. However, we have decreased $\max(|pq|, |rs|)$ which contradicts its minimality by assumption. Therefore, $|qr| = |pq| = |rs|$. By Observation A, $pqrs$ must be a regular trapezoid with $|ps| > |qr| = |pq| = |rs|$. See Figure 9. Now, we can rotate pq about x clockwise and rs about y counter-clockwisely by some amount to decrease $|pq|$ and $|rs|$, while maintaining that $|ps| > \max(|pq|, |rs|)$. Then we can switch the roles of qr and ps to obtain the four tuple $(r, s, p, q) \in \Phi(x, y)$ with a smaller $\max(|pq|, |rs|)$. This contradicts our assumption. In all, we conclude that w does not lie inside $pqrs$ or on pq or on rs . So w either lies outside $pqrs$ or on qr .

Suppose that w lies on qr . Then qr must be horizontal in order that $\max(|pq|, |rs|)$ is

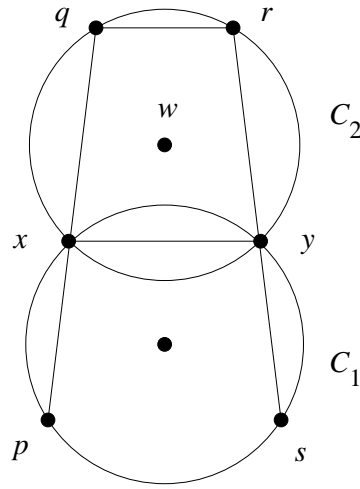


Figure 9.

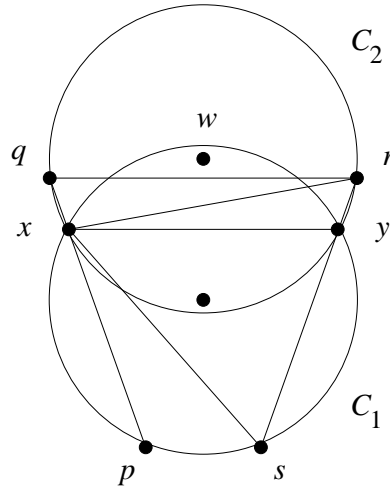


Figure 10.

minimized. At this position, $|pq| = |rs|$. Since we have proved before that $|qr| = |pq|$, we conclude that $|qr| = |pq| = |rs|$. It is clear that both $\angle qxy$ and $\angle ryx$ are obtuse at this position.

Suppose that w lies outside $pqrs$. Assume to the contrary that $|qr| > |rs|$. Observe that $\angle ryx$ and $\angle qxy$ are obtuse; otherwise, we can rotate rs about y counter-clockwise (resp. rotate pq about x clockwise) to decrease $|rs|$ (resp. decrease $|pq|$) and increase $|qr|$. This contradicts the minimality of $|pq| + |rs|$. We rotate rs about y counter-clockwise by an infinitesimal amount to increase $|qr|$ and $|rs|$ ($|pq|$ remains unchanged) such that $|qr| > |pq| > |rs|$. Now, we can rotate pq about x counter-clockwise by an infinitesimal amount to decrease $|qr|$ and $|pq|$ such that $|qr| > |pq| > |rs|$. But we have decreased $\max(|pq|, |rs|)$ and this contradicts our assumption. Hence, $|qr| = |pq| = |rs|$ and this completes the proof. \square

By Observation A and Lemma 5, we conclude that every element (p, q, r, s) in $\Phi^*(x, y)$ represents a regular trapezoid as shown in Figure 10.

3.2 Calculating β

Consider a $(p, q, r, s) \in \Phi^*(x, y)$. Let $\angle pqr = \theta$. By applying the sine law to triangle qrx and rsx , we obtain the equalities $|rx|/\sin \theta = |qr|/\sin(2\theta - \alpha_{xy})$ and $|rx|/\sin \alpha_{xy} = |rs|/\sin 2\alpha_{xy}$. By eliminating $|rx|$ from the above equations and canceling $|qr|$ and $|rs|$, we obtain $2 \sin \theta \cos \alpha_{xy} = \sin(2\theta - \alpha_{xy})$. By rearranging terms, we get

$$\tan \alpha_{xy} = \frac{2 \sin \theta (\cos \theta - 1)}{\cos 2\theta}. \quad (1)$$

For a fixed α_{xy} , we can solve Equation 1 for the smallest positive θ . This corresponds to minimizing $\max(|pq|, |rs|)$ and minimizing $|pq| + |rs|$. Thus, $\Phi^*(x, y)$ is a singleton set.

Our goal is to find the smallest α_{xy} such that $\Phi(x, y) \neq \emptyset$. Therefore, we differentiate Equation 1 with respect to θ and set $d(\alpha_{xy})/d\theta = 0$ to obtain $\cos \theta \cos \alpha_{xy} = \cos(2\theta - \alpha_{xy})$. By rearranging terms, we get

$$\tan \alpha_{xy} = \frac{\cos \theta - \cos 2\theta}{\sin 2\theta}. \quad (2)$$

By equating Equations 1 and 2, we obtain

$$\begin{aligned} & 2 \sin \theta (\cos \theta - 1) \sin 2\theta = (\cos \theta - \cos 2\theta) \cos 2\theta \\ \implies & 4(1 - \cos^2 \theta) \cos \theta (\cos \theta - 1) = \\ & (2 \cos^2 \theta - 1)(\cos \theta - 2 \cos^2 \theta + 1) \\ \implies & 2 \cos^2 \theta + 2 \cos \theta - 1 = 0 \\ \implies & \cos \theta = \frac{\sqrt{3}-1}{2} \text{ as } \cos \theta > 0. \end{aligned}$$

Substituting $\cos \theta = (\sqrt{3} - 1)/2$ into Equation 1, we obtain $\alpha_{xy} = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$. The corresponding β value is slightly less than 1.17682. Thus, we conclude that for any $\alpha_{xy} \geq \tan^{-1}(3/\sqrt{2\sqrt{3}})$, $\Phi(x, y) \neq \emptyset$. Conversely, the Remote Length Lemma is true for any $\alpha_{xy} < \tan^{-1}(3/\sqrt{2\sqrt{3}})$. This completes the proof of our main result.

Theorem 1 *Given a set S of points in the plane, the β -skeleton of S is a subgraph of a minimum weight triangulation of S for any $\beta > 1/\sin(\tan^{-1}(3/\sqrt{2\sqrt{3}}))$.*

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