On β-skeleton as a Subgraph of the Minimum Weight Triangulation

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Abstract

Given a set S of n points in the plane, a triangulation is a maximal set of non-intersecting edges connecting the points in S. The weight of the triangulation is the sum of the lengths of the edges. In this paper, we show that for $\beta > 1/\sin\kappa$, the β-skeleton of S is a subgraph of a minimum weight triangulation of S, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$. There exists a four point example such that the β-skeleton for $\beta < 1/\sin(\pi/3)$ is not a subgraph of the minimum weight triangulation.

Keywords: Minimum weight triangulation, beta skeleton, computational geometry.

1 Introduction

Let S be a set of n points in the plane. A *triangulation* $T(S)$ of S is a maximal set of non-intersecting straight line edges connecting points in S. Let $CH(S)$ denote the set of edges bounding the convex hull of S. Then $|T(S)| = 3n - 3 - |CH(S)|$ [6]. The *length* of an edge in $T(S)$ is equal to the Euclidean distance between its two endpoints. The *weight* of $T(S)$ is the sum of the lengths of edges in $T(S)$. The *minimum weight triangulation problem* is to compute $T(S)$ with minimum weight for a given point set S. The problem finds applications in numerical analysis [5, 8, 18]. However, the complexity of the problem remains open.

Several heuristics have been proposed to obtain a triangulation to approximate the MWT [4, 9, 12, 13, 14]. The heuristic in [14] is known to have a bound of $O(\log n)$ on the approximation ratio in the worst case. The more recently discovered heuristic [12] computes in $O(n \log n)$ time a triangulation with constant approximation ratio. Relatively little is known about the structure of the MWT. It is shown in [7] that the shortest edge between two points in S belongs to any MWT. Mark Keil [10] proves that a much larger graph, $\sqrt{2}$-skeleton, is always a subgraph of a

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MWT. The $\sqrt{2}$-skeleton is the β-skeleton defined by Kirkpatrick and Radke in [11] for $\beta = \sqrt{2}$. Given two points x and y, define xy to be the edge connecting x and y and define $|xy|$ to be the length of xy. For $\beta \geq 1$, the *forbidden neighborhood* of x and y is the union of two disks with radius $\beta|xy|/2$ that pass through both x and y. Given a point set S and $x, y \in S$, xy belongs to the β-skeleton of S if no point in S lies in the interior of the forbidden neighborhood of x and y. Refer to Figure 1. Let α_{xy} be the angle that the chord xy subtends at one of the circles. Then $\beta = 1/\sin\alpha_{xy}$.

It is conjectured in [10] that the β-skeleton is a subgraph of a MWT for $\beta \geq 1/\sin(\pi/3)$. Recently, it is reported in [16] that the value of β can be improved to $1/\sin(2\pi/7) \approx 1.279$. Yang *et al.* [17] formulated and proved a different property: if the union of the two disks centered at x and y with radius $|xy|$ is empty, then xy is in a MWT (this interpretation of the original statement in [17] is from [1]). Note that the subgraph generated by the above condition and the β-skeleton do not contain each other for $\beta > 1/\sin(\pi/3)$, but for $\beta \leq 1/\sin(\pi/3)$, the β-skeleton contains the subgraph generated by the above condition.

In this paper, we show that the β-skeleton is a subgraph of a MWT, for $\beta > 1/\sin\kappa \approx 1.17682$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$. Both our result and the result in [16] are based on proving an improved version of the key lemma, Remote Length Lemma, in [10]. Moreover, the proof strategy in [10] cannot be pushed further to improve upon our result. There exists a four point example such that the β-skeleton for $\beta < 1/\sin(\pi/3) \approx 1.1547$ is not a subgraph of any MWT. Refer to Figure 1. The two circles define the forbidden region for xy for $\beta_0 = 1/\sin(\pi/3)$. The triangle axy is equilateral. The two shaded disks define the forbidden region for xy for $\beta_1 < 1/\sin(\pi/3)$. Thus, $bx < ax = xy$. We can pick a point c on the boundary of the lower shaded disk such that $bc < xy$. So xy belongs to the β_1-skeleton of $\{b, c, x, y\}$ but the MWT of $\{b, c, x, y\}$ contains bc instead of xy. After the appearance of a preliminary version of this paper [3], it has been proved recently [15] that $1/\sin\kappa$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}})$, is indeed a lower bound on β for β-skeleton to be a subgraph of a MWT.

In Section 2, we shall review Keil's proof. Our result is presented in Section 3.

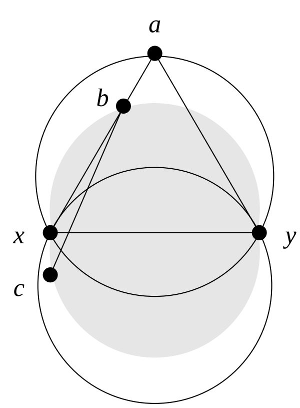

Figure 1.

2 Preliminaries

Keil's proof follows the edge insertion paradigm [2]. Assume to the contrary that xy is an edge of a β-skeleton that does not belong to a MWT $\mathcal{T}$. The strategy is to add xy to $\mathcal{T}$ and remove the existing edges that intersect xy. Then the two resulting polygonal regions on both sides of xy are retriangulated carefully to obtain a new triangulation. A contradiction is derived by arguing that the new triangulation has a smaller weight than $\mathcal{T}$. We describe the main ideas below. Assume throughout that $\alpha_{xy} < \pi/3$.

Let e_j, $1 \leq j \leq m$, be the edges intersected by xy and let $|e_{j-1}| \leq |e_j|$, $2 \leq j \leq m$. Let P be the polygonal region above xy to be retriangulated incrementally. (The polygonal region below xy can be dealt with similarly.) During the incremental retriangulation, we shall obtain a sequence of triangulated polygons P_j, $0 \leq j \leq m$, such that P_0 is the degenerate polygon xy, P_m is a triangulation of P, and $P_{j-1} \subseteq P_j$. P_j is obtained from P_{j-1} by expanding P_{j-1} to include the endpoint v_j of e_j as follows (v_j is the endpoint on the same side of xy as P). If v_j lies in P_{j-1}, then $P_j = P_{j-1}$. Otherwise, e_j intersects a boundary edge v_iv_k of P_{j-1}. In general, the triangle $v_iv_jv_k$ contains a subsequence σ_1 of vertices on P from v_i to v_j and another subsequence σ_2 from v_j to v_k. See Figure 2: the polygon with solid boundary is P_{j-1}, the bold triangle is $v_iv_jv_k$, the polygon with dashed boundary is P, the white dots inside the bold triangle is σ_1, and the grey dots inside the bold triangle is σ_2. We arbitrarily triangulate the polygon $v_i\sigma_1v_j\sigma_2v_k$ and P_j is the union of this triangulated polygon and P_{j-1}. We claim that all the new edges added are shorter than e_j. Thus, we shall inductively obtain a new triangulation of lesser weight than $\mathcal{T}$ (and so the contradiction).

The proof of the claim is as follows. All new edges added have length at most $\max\{|v_iv_j|, |v_jv_k|, |v_iv_k|\}$. v_iv_k is shorter than e_{j-1} by induction assumption. Consider v_iv_j (v_jv_k can be handled similarly). If v_i lies in triangle xv_jy, then by triangle inequality and the fact that $\alpha_{xy} < \pi/3$, v_iv_j is shorter than e_j. Otherwise, consider the convex hull of the chain from x to v_j on P_j. v_i must lie in a triangle $v_av_bv_j$, where v_a and v_b are hull vertices. Thus, $|v_iv_j| \leq \max\{|v_av_j|, |v_av_b|, |v_bv_j|\}$. Since v_a and v_b are hull vertices, v_a and v_b were added in the growth process in the past. Thus, the edges e_a and e_b, with endpoints v_a and v_b respectively, were processed before e_j. So $|e_a| \leq |e_j|$ and $|e_b| \leq |e_j|$. Applying the following Lemma 1 to v_av_j implies that $|v_av_j| < |e_j|$. Similarly, we obtain $|v_bv_j| < |e_j|$ and $|v_av_b| < |e_j|$. Thus, $|v_iv_j| < |e_j|$ and this completes the proof. Refer to Figure 3 for an illustration of the Remote Length Lemma.

Lemma 1 (Remote Length Lemma) *[10] Suppose that $\beta \geq \sqrt{2}$. Let x and y be the endpoints of an edge in the β-skeleton of a set S of points in the plane. Let p, q, r, and s be four other distinct points of S such that pq intersects the interior of xy, rs intersects the interior of xy, pq and rs does not intersect the interior of each other and p and s lie on the same side of the line through xy. Then either $|qr| < |pq|$ or $|qr| < |rs|$.*

As observed in [10], for $1/\sin(\pi/3) \leq \beta < \sqrt{2}$, the only part of the entire proof in [10] that may fail is the Remote Length Lemma. We achieve our result by showing that the Remote Length Lemma is true for $\beta > 1/\sin\kappa \approx 1.17682$, where $\kappa = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$.

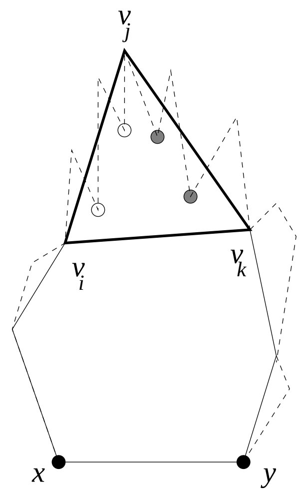

Figure 2.

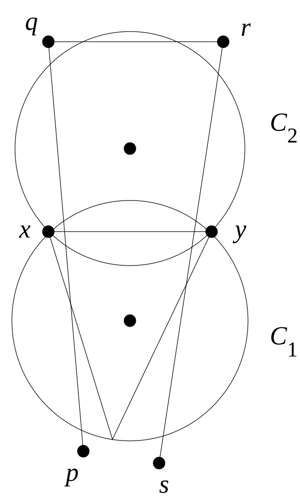

Figure 3.

3 The proof

Let x and y be the endpoints of an edge in the β-skeleton of a set S of points in the plane. Let (p, q, r, s) be a four tuple of distinct points (not necessarily in S) outside or on the boundary of the forbidden neighborhood of xy, such that pq intersects xy, rs intersects xy, pq and rs does not intersect the interior of each other and p and s lie on the same side of the line through xy. If $|qr| \geq |pq|$ and $|qr| \geq |rs|$, then we say that (p, q, r, s) satisfies the *remote length exception* with respect to xy. Refer to Figure 3. Let the two circles be C_1 and C_2. Throughout this paper, we assume that α_{xy} is some fixed constant such that $\alpha_{xy} < \pi/3$ and there exists some (p, q, r, s) that satisfies the remote length exception with respect to xy.

Define $\Phi(x, y)$ be the set of four tuples of points (p, q, r, s) such that (p, q, r, s) satisfies the remote length exception with respect to xy. The basic idea of our proof is to compute the smallest value κ for α_{xy} such that $\Phi(x, y) \neq \emptyset$. In other words, for all values of $\alpha_{xy} < \kappa$, $\Phi(x, y) = \emptyset$ and therefore, the Remote Length Lemma holds in general. The corresponding value, $1/\sin\kappa$, for β will give us an improvement upon the result in [10].

Since there can be an infinite number of four tuples (p, q, r, s) that belong to $\Phi(x, y)$, it is not clear how to compute κ and hence β directly. Instead, we restrict our attention to a critical structure that must exist in $\Phi(x, y)$ if $\Phi(x, y) \neq \emptyset$. We first fully characterize this critical structure. Select a subset $\mathcal{A} = \{(p, q, r, s) \in \Phi(x, y) : \max(|pq|, |rs|) \text{ is minimized } \}$. Then select a subset $\Phi^*(x, y) = \{(p, q, r, s) \in \mathcal{A} : |pq| + |rs| \text{ is minimized } \}$. $\Phi^*(x, y)$ turns out to be a singleton set containing this critical structure. Then, we compute κ based on this knowledge. The characterization of the critical structure is given in the next section. The calculation of κ and β is given in Section 3.2.

3.1 Characterizing $\Phi^*(x, y)$

The main result in this section is that if $(p, q, r, s) \in \Phi^*(x, y)$, then $|qr| = |pq| = |rs|$, $\angle qxy = \angle ryx$ and they are obtuse. See Figure 4. There are several geometric facts Observation A, Observation B, and Observation C that we will use in our argument. Observation A refers to Figure 5(a), Observation B refers to Figure 5(b) and Observation C refers to Figure 5(c).

Observation A *Let cd be a line segment through x with endpoints on C_1 and C_2. Then $|cd|$ is a continuous concave function F in $\angle cxy$. Moreover, the slope of F becomes zero only when $\angle cxy = \pi/2$, F is symmetric around $\angle cxy = \pi/2$, and $|cd|$ is maximized when $\angle cxy = \pi/2$.*

Observation B *Let ef be a line segment with endpoints e on C_1 and f on C_2 such that the two centers of C_1 and C_2 lie on the same side of ef and ef intersects the interior of xy. If f (resp. e) slides on C_2 (resp. C_1) such that ef rotates away from the centers and ef still intersects xy, then $|ef|$ decreases.*

Observation C *Let ef be a line segment with endpoints e on C_1 and f on C_2 such that the two centers of C_1 and C_2 lie on opposite sides of ef and ef intersects the interior of xy. If f is closer to y (resp. x), then sliding f along C_2 clockwisely (resp. counter-clockwisely) decreases $|ef|$, provided that ef still intersects xy. If e is closer to x (resp. y), then sliding e along C_1*

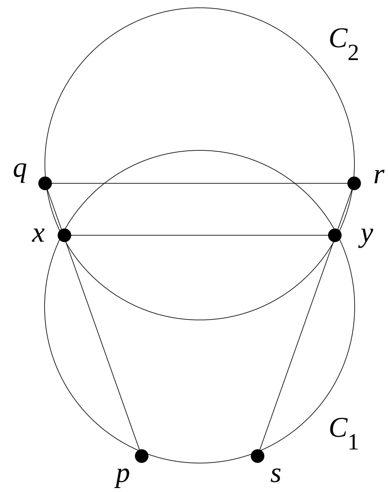

Figure 4.

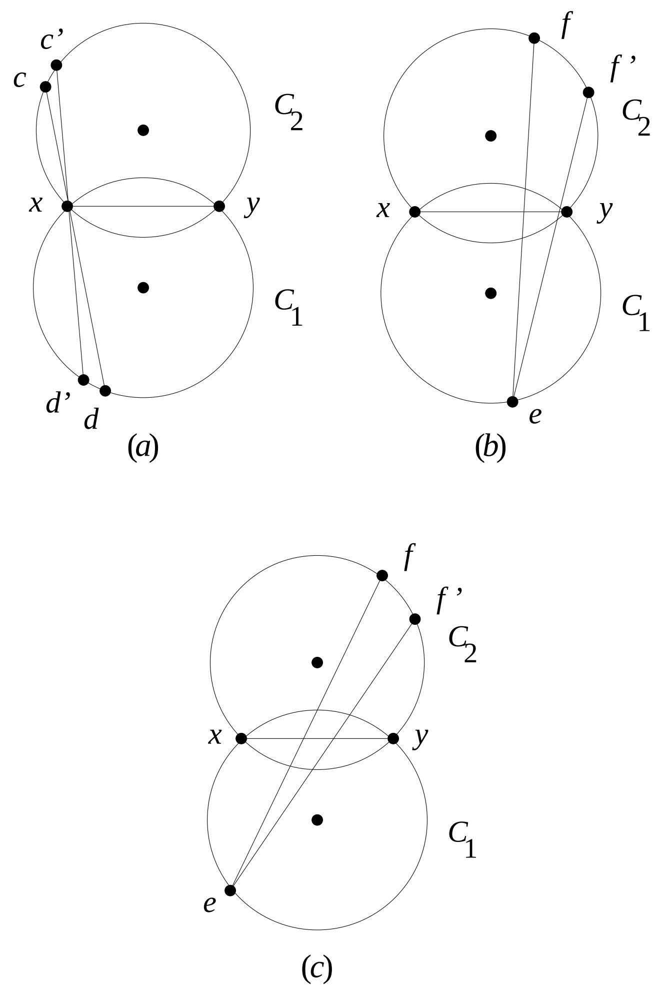

Figure 5.

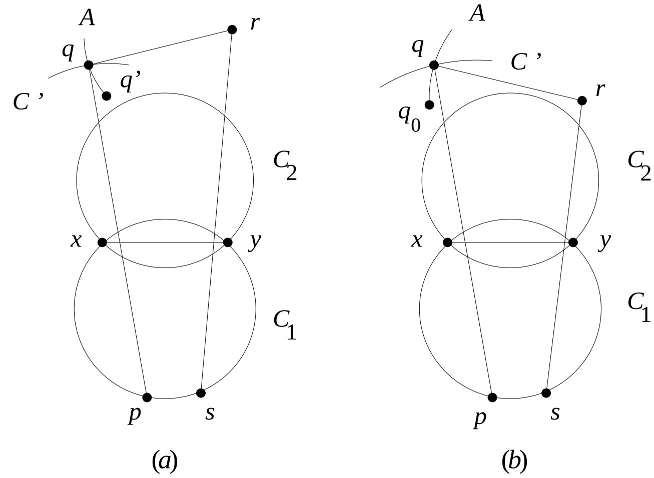

Figure 6.

clockwisely (resp. counter-clockwisely) decreases $|ef|$, provided that ef still intersects xy.

Lemma 2 *If $(p,q,r,s) \in \Phi^*(x,y)$, then p and s lie on C_1, $p \neq s$, and q and r lie on C_2.*

Proof Refer to Figure 3. If p does not lie on C_1, then we can shorten pq to make p lie on C_1. This contradicts that $|pq|+|rs|$ is minimized. The same argument holds for s. So p and s lie on C_1. Assume to the contrary that $p = s$. Then qr is the longest side of the triangle pqr, which implies that $\angle qpr \geq \pi/3$. However, $\angle xpy \geq \angle qpr \geq \pi/3$ which contradicts our assumption that $\alpha_{xy} = \angle xpy < \pi/3$. In the following, assume to the contrary that q does not lie on C_2. The treatment for r is similar.

Case(1) : $\angle rqp \geq \pi/2$. Refer to Figure 6(a). Let C' be the circle with center p and radius $|pq|$. Draw a circular arc A through q with center r and radius $|qr|$ such that A does not intersect C_2 or rs and A intersects C' exactly once at q. The endpoint q' of A shown in the figure must lie inside C' but outside C_2. Thus $|q'r| = |qr|$, $\max(|pq'|,|rs|) \leq \max(|pq|,|rs|)$, but $|pq'| < |pq|$. Hence, $(p,q',r,s) \in \Phi(x,y)$ and $|pq'|+|rs| < |pq|+|rs|$. This contradicts our assumption that $|pq|+|rs|$ is the minimum possible.

Case(2) : $\angle rqp < \pi/2$. Refer to Figure 6(b). Let C' be the circle with center p and radius $|pq|$. Draw a circular arc A through q with center r and radius $|qr|$ such that A does not intersect C_2 or rs and A intersects C' exactly once at q. The endpoint q_0 of A shown in the figure must lie outside the quadrilateral $pqrs$ and C_2 but inside C'. If pq does not pass through x, then A can be made short enough such that pq_0 intersects xy. Then $(p,q_0,r,s) \in \Phi(x,y)$ and $|pq_0| < |pq|$ which contradicts the minimality of $|pq|+|rs|$. Suppose that pq passes through x. Draw a line segment from q_0 through x to p_0 on C_1. Let the other endpoint of A be q_1. Draw another line segment from q_1 through x to p_1 on C_1. Denote by B the circular arc on C_1 traversed clockwisely from p_0 to p_1. For an arbitrary point q_t on A, define p_t to be the point on B such that p_tq_t passes through x. See Figures 7. Let $\theta_0 = \angle q_0xy$, $\theta_1 = \angle q_1xy$, and $c = \angle rxy$. Let $\theta^* = \angle qxy$ and $\theta = \angle q_txy$. Then

$$\begin{aligned} |q_tx| &= |rx|\cos(\theta - c) + \sqrt{|q_tr|^2 - |rx|^2\sin^2(\theta - c)} \\ |p_tx| &= |xy|\sin(\theta - \alpha_{xy})/\sin\alpha_{xy}. \end{aligned}$$

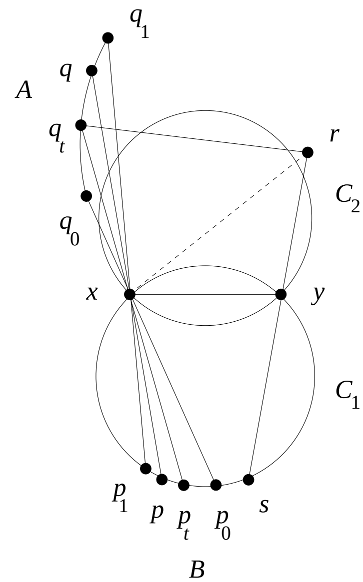

Figure 7.

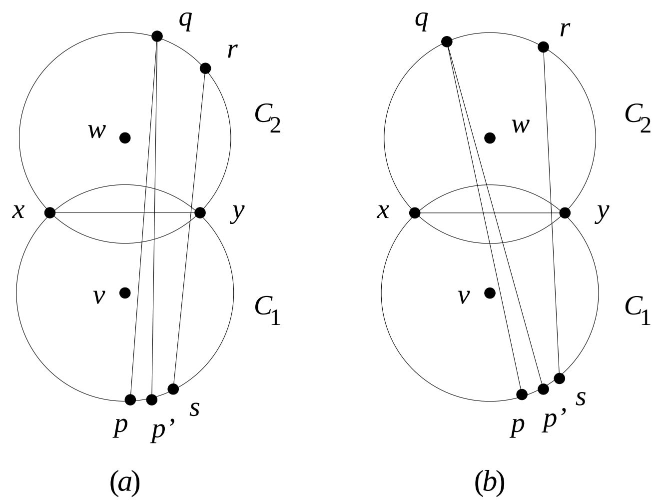

Figure 8.

It is clear from the figure that both $|q_t x|$ and $|p_t x|$ are concave in $[\theta_1, \theta_0]$. Moreover, since $|q_t x|$ and $|p_t x|$ are trigonometric, they are concave functions with a unique maximum in $[\theta_1, \theta_0]$. Therefore, within $[\theta_1, \theta_0]$, $|p_t q_t| = |p_t x| + |q_t x|$ must have at most one stationary point (the unique maximum if it exists) and $|p_t q_t|$ achieves the minimum at θ_0 or θ_1 or both. Since $\theta^* \in (\theta_1, \theta_0)$, we conclude that $|p_0 q_0| < |pq|$ or $|p_1 q_1| < |pq|$. So $(p_0, q_0, r, s) \in \Phi(x, y)$ or $(p_1, q_1, r, s) \in \Phi(x, y)$ and this contradicts the minimality of $|pq| + |rs|$. $\square$

Lemma 3 *Let v and w be the centers of C_1 and C_2. If $(p, q, r, s) \in \Phi^*(x, y)$, then v and w lie on the right of pq and on the left of rs, respectively.*

Proof By Lemma 2, p and s lie on C_1 and q and r lie on C_2. Assume to the contrary that the lemma is not true. Then either v and w lie on the same side of pq and rs (Case(1)), or v and w lie on opposite sides of pq or rs (Case(2)).

Case(1) : Assume without loss of generality that v and w lie on the left of pq and rs. Refer to Figure 8(a). Since rs lies to the right of pq, pq does not pass through y. Since $p \neq s$, by Observation B, we can slide p along C_1 counter-clockwisely to decrease $|pq|$, but this contradicts the minimality of $|pq| + |rs|$.

Case(2) : Assume without loss of generality that v and w lie on opposite sides of pq. Refer to Figure 8(b). By Observation C, we can slide p along C_1 either clockwisely or counter-clockwisely to decrease $|pq|$, depending on whether p is closer to x or y. This contradicts the minimality of $|pq| + |rs|$. $\square$

Lemma 4 *If $(p, q, r, s) \in \Phi^*(x, y)$, then pq passes through x and rs passes through y.*

Proof First, (p, q, r, s) satisfies Lemma 2 and Lemma 3. If pq (resp. rs) does not pass through x (resp. y), then by Observation B, we can slide p along C_1 clockwisely (resp. s along C_1 counter-clockwisely) and decrease $|pq|$ (resp. $|rs|$). This contradicts the minimality of $|pq| + |rs|$. $\square$

Lemma 5 *If $(p, q, r, s) \in \Phi^*(x, y)$, then $|qr| = |pq| = |rs|$ and $\angle qxy$ and $\angle ryx$ are obtuse.*

Proof First, (p, q, r, s) satisfies Lemmas 2–4. Since $(p, q, r, s) \in \Phi^*(x, y)$, $|qr| \geq \max(|pq|, |rs|)$. Without loss of generality, assume that $|pq| = \max(|pq|, |rs|)$. Let w be the center of C_2. For brevity, *rotating pq about x or rs about y* means that we keep p and s on C_1 and q and r on C_2 during the rotation.

Assume to the contrary that $|qr| > |pq|$. If $\angle pxy \leq \pi/2$, then we can rotate pq about x counter-clockwisely by an infinitesimal amount and still maintain that $|qr| > \max(|pq|, |rs|)$. However, by Observation A, $|pq|$ decreases which contradicts the minimality of $|pq| + |rs|$. If $\angle pxy > \pi/2$, then $\angle qxy < \pi/2$. We can rotate pq about x clockwisely by an infinitesimal amount and still maintain that $|qr| > \max(|pq|, |rs|)$. By Observation A, $|pq|$ decreases which contradicts the minimality of $|pq| + |rs|$. Hence, we conclude that $|qr| = |pq|$.

We claim that w does not lie inside the quadrilateral $pqrs$ or on pq or on rs. Assume to the contrary this is not true. Observe that $\angle ryx < \pi/2$; otherwise, we can rotate rs about y clockwisely to increase $|qr|$ and decrease $|rs|$, which contradicts the minimality of $|pq| + |rs|$. By a similar argument, $\angle qxy$ must also be acute. If $|qr| = |pq| > |rs|$, then we can rotate rs about y clockwisely by an infinitesimal amount to increase $|qr|$ and $|rs|$ ($|pq|$ remains unchanged) such that $|qr| > |pq| > |rs|$. But then we can rotate pq about x clockwisely by an infinitesimal amount to decrease $|qr|$ and $|pq|$ such that $|qr| > |pq| > |rs|$. However, we have decreased $\max(|pq|, |rs|)$ which contradicts its minimality by assumption. Therefore, $|qr| = |pq| = |rs|$. By Observation A, $pqrs$ must be a regular trapezoid with $|ps| > |qr| = |pq| = |rs|$. See Figure 9. Now, we can rotate pq about x clockwisely and rs about y counter-clockwisely by some amount to decrease $|pq|$ and $|rs|$, while maintaining that $|ps| > \max(|pq|, |rs|)$. Then we can switch the roles of qr and ps to obtain the four tuple $(r, s, p, q) \in \Phi(x, y)$ with a smaller $\max(|pq|, |rs|)$. This contradicts our assumption. In all, we conclude that w does not lie inside $pqrs$ or on pq or on rs. So w either lies outside $pqrs$ or on qr.

Suppose that w lies on qr. Then qr must be horizontal in order that $\max(|pq|, |rs|)$ is

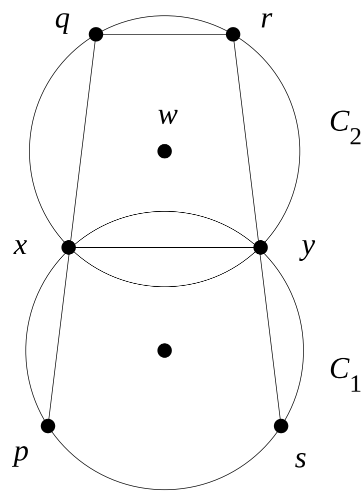

Figure 9.

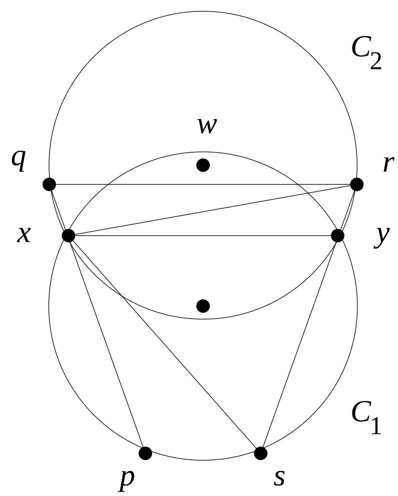

Figure 10.

minimized. At this position, $|pq| = |rs|$. Since we have proved before that $|qr| = |pq|$, we conclude that $|qr| = |pq| = |rs|$. It is clear that both $\angle qxy$ and $\angle ryx$ are obtuse at this position.

Suppose that w lies outside $pqrs$. Assume to the contrary that $|qr| > |rs|$. Observe that $\angle ryx$ and $\angle qxy$ are obtuse; otherwise, we can rotate rs about y counter-clockwisely (resp. rotate pq about x clockwisely) to decrease $|rs|$ (resp. decrease $|pq|$) and increase $|qr|$. This contradicts the minimality of $|pq| + |rs|$. We rotate rs about y counter-clockwisely by an infinitesimal amount to increase $|qr|$ and $|rs|$ ($|pq|$ remains unchanged) such that $|qr| > |pq| > |rs|$. Now, we can rotate pq about x counter-clockwisely by an infinitesimal amount to decrease $|qr|$ and $|pq|$ such that $|qr| > |pq| > |rs|$. But we have decreased $\max(|pq|, |rs|)$ and this contradicts our assumption. Hence, $|qr| = |pq| = |rs|$ and this completes the proof. $\square$

By Observation A and Lemma 5, we conclude that every element (p, q, r, s) in $\Phi^*(x, y)$ represents a regular trapezoid as shown in Figure 10.

3.2 Calculating β

Consider a $(p, q, r, s) \in \Phi^*(x, y)$. Let $\angle pqr = \theta$. By applying the sine law to triangle qrx and rsx, we obtain the equalities $|rx|/\sin\theta = |qr|/\sin(2\theta - \alpha_{xy})$ and $|rx|/\sin\alpha_{xy} = |rs|/\sin 2\alpha_{xy}$. By eliminating $|rx|$ from the above equations and canceling $|qr|$ and $|rs|$, we obtain $2\sin\theta\cos\alpha_{xy} = \sin(2\theta - \alpha_{xy})$. By rearranging terms, we get

$$\tan\alpha_{xy} = \frac{2\sin\theta(\cos\theta - 1)}{\cos 2\theta}. \tag{1}$$

For a fixed α_{xy}, we can solve Equation 1 for the the smallest positive θ. This corresponds to minimizing $\max(|pq|, |rs|)$ and minimizing $|pq| + |rs|$. Thus, $\Phi^*(x, y)$ is a singleton set.

Our goal is to find the smallest α_{xy} such that $\Phi(x, y) \neq \emptyset$. Therefore, we differentiate Equation 1 with respect to θ and set $d(\alpha_{xy})/d\theta = 0$ to obtain $\cos\theta\cos\alpha_{xy} = \cos(2\theta - \alpha_{xy})$. By rearranging terms, we get

$$\tan\alpha_{xy} = \frac{\cos\theta - \cos 2\theta}{\sin 2\theta}. \tag{2}$$

By equating Equations 1 and 2, we obtain

$$\begin{aligned}
& 2\sin\theta(\cos\theta - 1)\sin 2\theta = (\cos\theta - \cos 2\theta)\cos 2\theta \\
\Longrightarrow\ & 4(1 - \cos^2\theta)\cos\theta(\cos\theta - 1) = \\
& (2\cos^2\theta - 1)(\cos\theta - 2\cos^2\theta + 1) \\
\Longrightarrow\ & 2\cos^2\theta + 2\cos\theta - 1 = 0 \\
\Longrightarrow\ & \cos\theta = \tfrac{\sqrt{3}-1}{2} \text{ as } \cos\theta > 0.
\end{aligned}$$

Substituting $\cos\theta = (\sqrt{3} - 1)/2$ into Equation 1, we obtain $\alpha_{xy} = \tan^{-1}(3/\sqrt{2\sqrt{3}}) \approx \pi/3.1$. The corresponding β value is slightly less than 1.17682. Thus, we conclude that for any $\alpha_{xy} \geq \tan^{-1}(3/\sqrt{2\sqrt{3}})$, $\Phi(x, y) \neq \emptyset$. Conversely, the Remote Length Lemma is true for any $\alpha_{xy} < \tan^{-1}(3/\sqrt{2\sqrt{3}})$. This completes the proof of our main result.

Theorem 1 *Given a set S of points in the plane, the β-skeleton of S is a subgraph of a minimum weight triangulation of S for any $\beta > 1/\sin(\tan^{-1}(3/\sqrt{2\sqrt{3}})$.*

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