k-Regret Minimizing Set

Efficient Algorithms and Hardness

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ICDT, 2017

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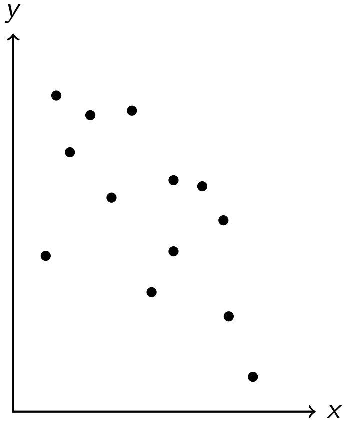

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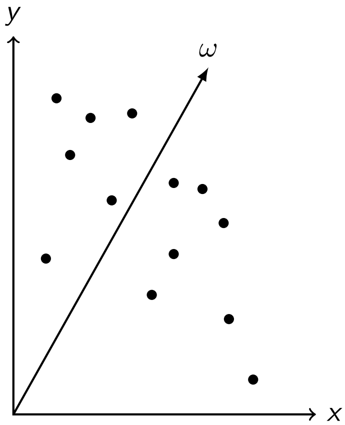

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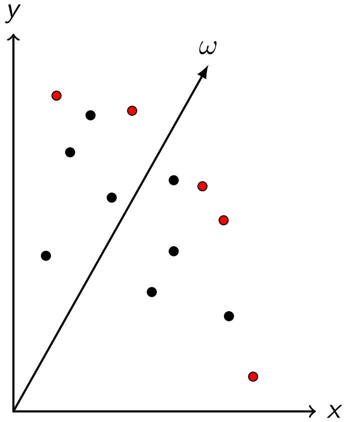

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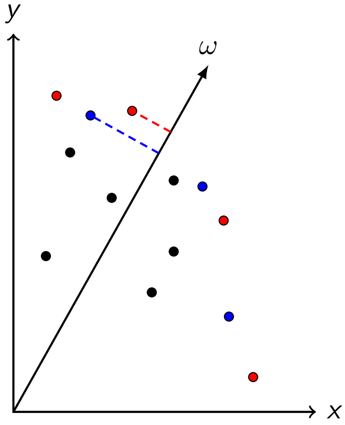

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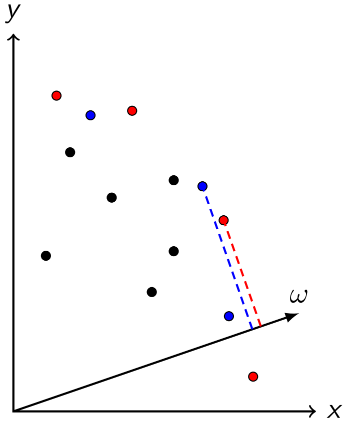

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[CTVW14]:

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- **Decision**:
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  - RMS in *expected* $O(n \log n)$.
- When $d \geq 3$, Dec-RMS is NP-hard.

Geometric View

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\begin{aligned}
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&\Downarrow \\
f_p(\lambda) &= \langle p, \omega \rangle \\
&= (1-\lambda)x + \lambda y.
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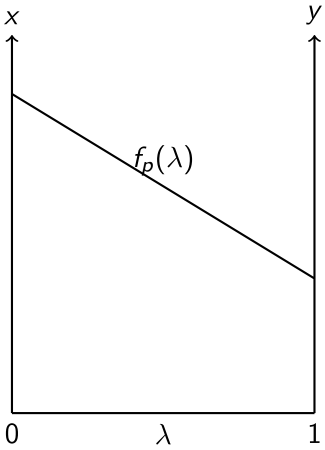

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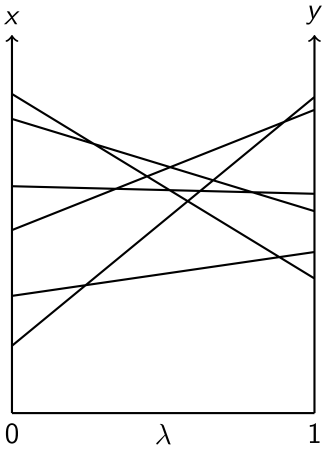

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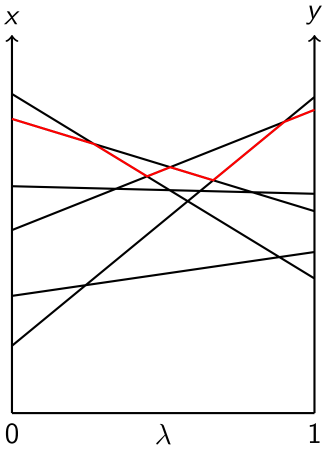

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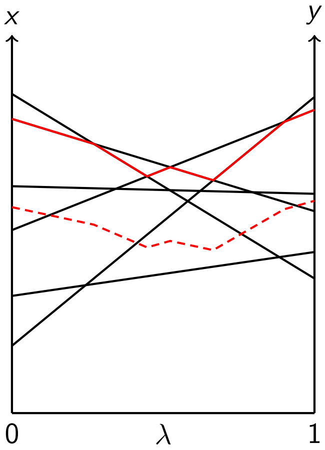

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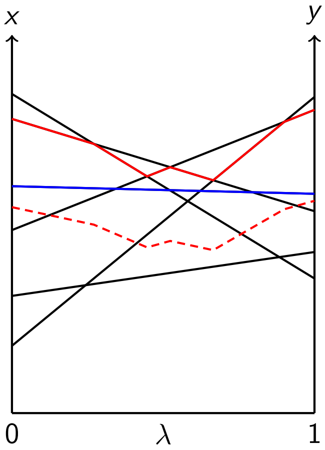
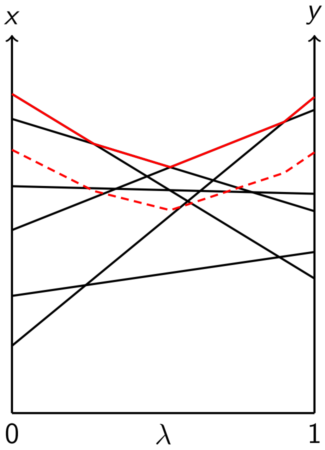
$$\max_{p \in R} f_p(\lambda) \geq \theta\text{-}LS_k(\lambda)$$



Warm-Up: Dec-RMS

- When $k = 1$, LS_1 (and thus θ-LS_1) is convex.



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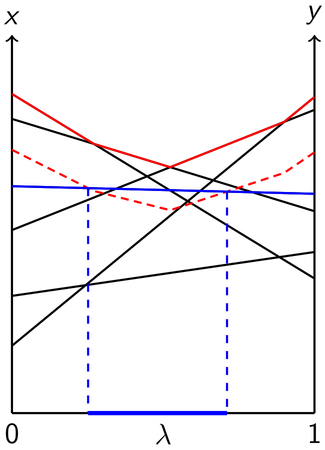

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- **Solution**:
  Greedily increment the initial covered interval $[0, b]$.

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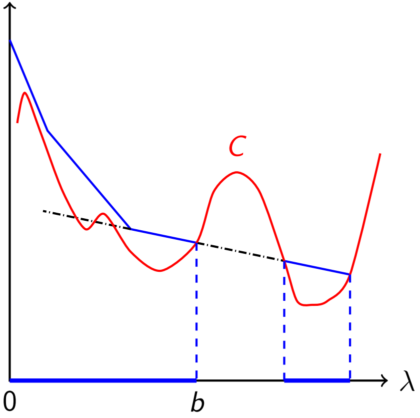

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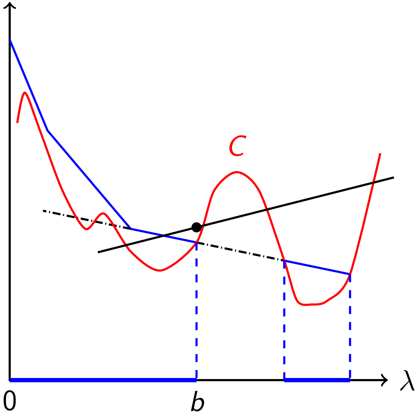

Dec-k-RMS

We apply the following greedy algorithm:

- Keep selected lines R as a stack;
- Once the new line increments the initial covered interval, push it into R;
- While pushing, pop out the redundant lines from R.

Dec-k-RMS (Example)

R :

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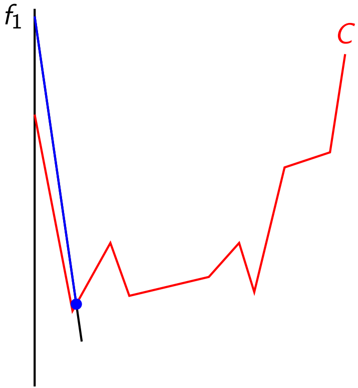

$R: \quad f_1$

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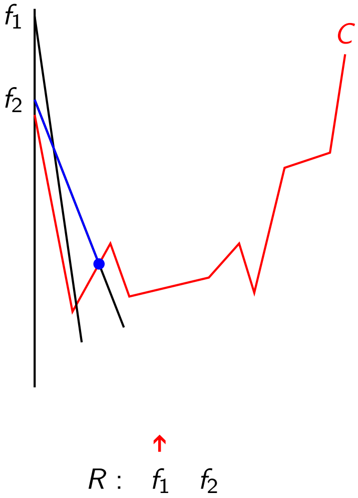

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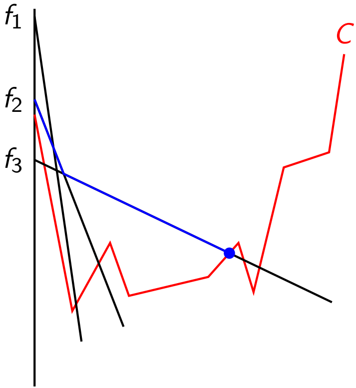

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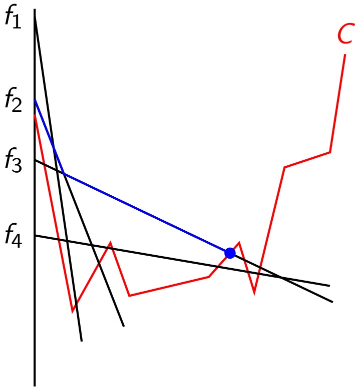

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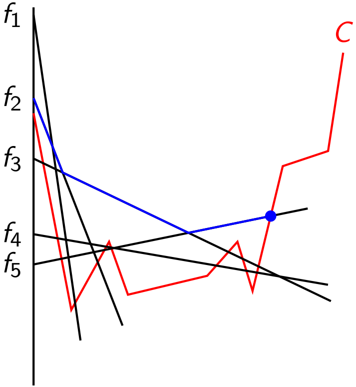

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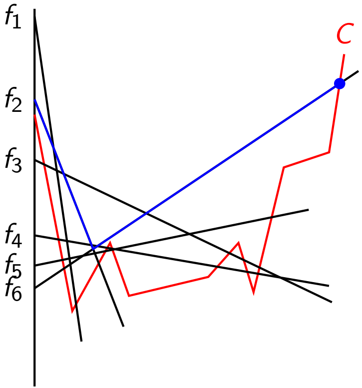

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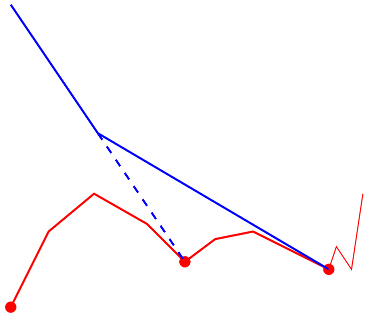

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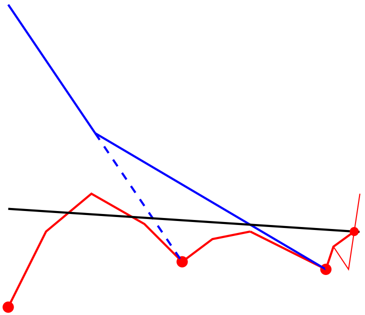

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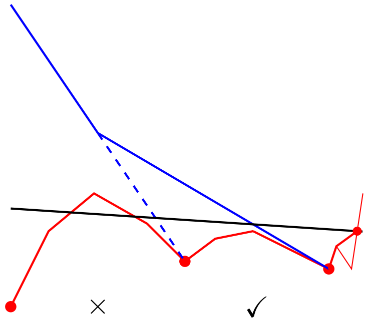

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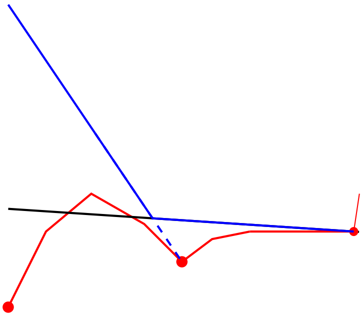

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- Maintain the stack of convex hulls: $O(n + m)$.

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- **Solution**: Implicitly store $|Cand(D)|$ using *sweep line* over $X(D)$.

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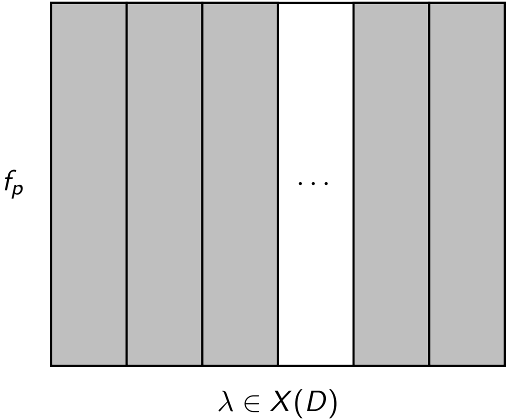

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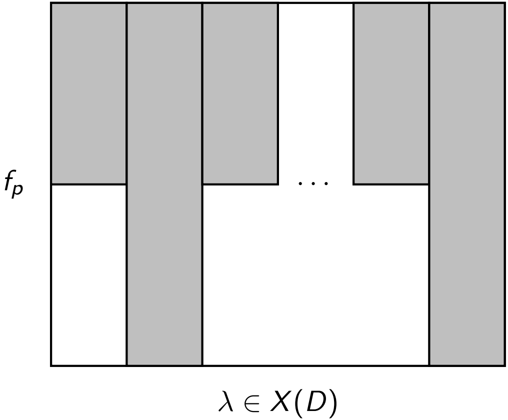

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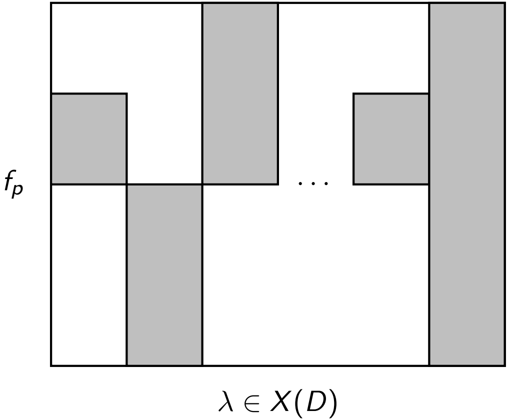

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- All possible values of θ:

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- $O(n \log n)$ time for n random sampling each round, expected $O(1)$ round.

RMS

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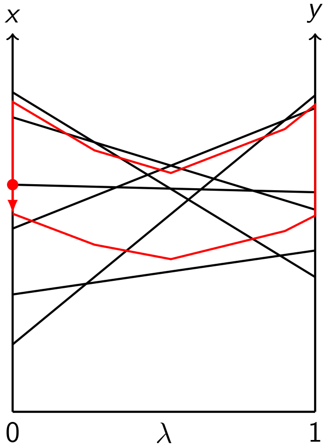

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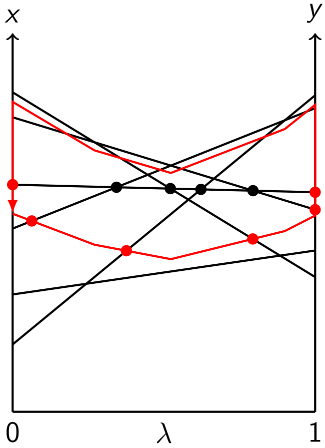

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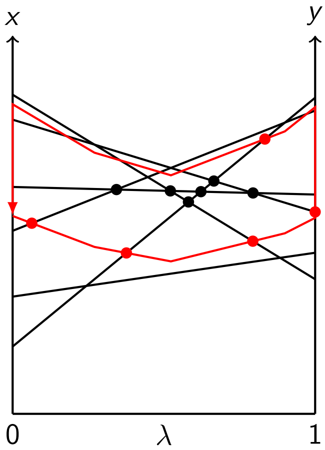

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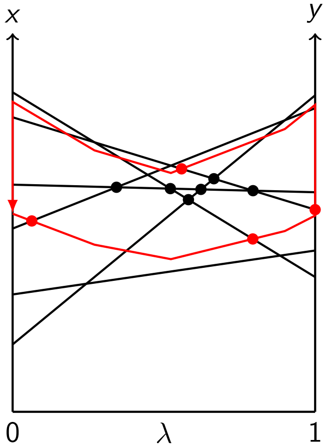

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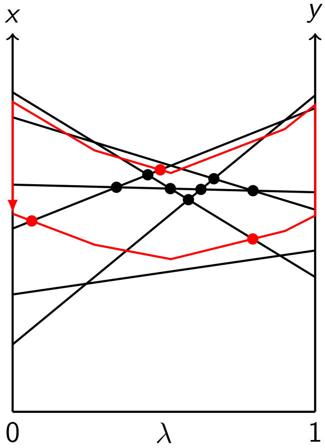

3D Dec-RMS is NP-hard

- **Observation**: If a spherical triangle $\triangle ABC$ contains the circumcenter P, then $\{A, B, C\}$ has fixed regret ratio in respect to P:

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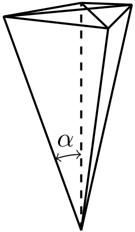

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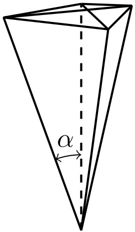
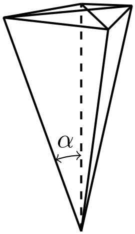
- So for $\theta \approx \cos\alpha$, one either choose P or choose ABC.



3D Dec-RMS is NP-hard

- **Observation**: If a spherical triangle $\triangle ABC$ contains the circumcenter P, then $\{A, B, C\}$ has fixed regret ratio in respect to P:
$$1 - \theta = 1 - \cos\alpha.$$
- So for $\theta \approx \cos\alpha$, one either choose P or choose ABC.
- Reduced to **Vertex Cover** on a highly constraint class of planar graphs, which we project to the sphere surface.



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