COMP 670J – Addendum

This page proves the statement

if $y \in P_i$ *with* $y \leq t$ *then* $\exists z \in L_i$ *such that*

$$\left(1 - \frac{\epsilon}{n}\right)^i y \leq z \leq y \tag{1}$$

that appeared in the analysis of the subset-sum approximation scheme presented in COMP670J, Fall 2001. Please see the class notes for definitions of the terms.

The proof will be by induction on i. When $i = 1$ note that $P_1 =< 0, x_1 >$ while $L_1 =< 0, x_1 >$ or $L_1 =< 0 >$, depending upon whether or not $x_i \leq t$. In both cases, if $y \in P_1$ with $y \leq t$ then $y \in L_1$ so (1) is true.

Now assume that (1) is true for i. We will prove its correctness for $i + 1$. Recall that $P_{i+1} = Merge(P_i, P_i + x_{i+1})$ with all items $> t$ thrown out and L_{i+1} is the trimmed version of $Merge(L_i, L_i + x_{i+1})$ with all items $> t$ thrown out.

So now suppose that $y \in P_{i+1}$ with $y \leq t$.
There are two cases: (i) $y \in P_i$ or (ii) $y \in P_i + x_{i+1}$.

If $y \in P_i$ then, by induction, $\exists z_i \in L_i$ such that $\left(1 - \frac{\epsilon}{n}\right)^i y \leq z_i \leq y$. Since $z_i \in Merge(L_i, L_i + x_{i+1})$, $\exists z \in L_{i+1}$ such that $\left(1 - \frac{\epsilon}{n}\right) z_i \leq z \leq z_i$. Combining the two sets of inequalities yields

$$\left(1 - \frac{\epsilon}{n}\right)^{i+1} y \leq \left(1 - \frac{\epsilon}{n}\right) z_i \leq z \leq z_i \leq y$$

which is what we wanted to show.

If $y \in P_i + x_{i+1}$ then $y = y_i + x_{i+1}$ for some $y_i \in P_i$. Again by induction $\exists z_i \in L_i$ such that $\left(1 - \frac{\epsilon}{n}\right)^i y_i \leq z_i \leq y_i$. Therefore

$$\left(1 - \frac{\epsilon}{n}\right)^i y = \left(1 - \frac{\epsilon}{n}\right)^i (y_i + x_{i+1}) \leq \left(1 - \frac{\epsilon}{n}\right)^i y_i + x_{i+1} \leq z_i + x_{i+1} \leq y_i + x_{i+1} = y.$$

Since $z_i + x_{i+1} \in Merge(L_i, L_i + x_{i+1})$, $\exists z \in L_{i+1}$ such that $\left(1 - \frac{\epsilon}{n}\right)(z_i + x_{i+1}) \leq z \leq z_i + x_{i+1}$. Combining the two sets of inequalities yields

$$\left(1 - \frac{\epsilon}{n}\right)^{i+1} y \leq \left(1 - \frac{\epsilon}{n}\right)(z_i + x_{i+1}) \leq z \leq z_i + x_{i+1} \leq y$$

which is what we wanted to show and the proof is now complete.