Lecture 5: The Multiway Cut Problem
(Vazirani (Chapter 4)))

Let $G = (V, E)$ be a weighted undirected graph.
Let $S \subseteq V$. The *cut* defined by S is the set of edges

$$\{(u, v) \,:\, u \in S,\, v \in V - S\}$$

that crosses the vertex partition $(S, V - S)$.
The *cost* of a cut is

$$\delta(S) = \sum_{\{(u,v)\,:\, u \in S,\, v \in V-S\}} w(u, v).$$

Note: The removal of the cut $(S, V - S)$ disconnects G.

Given *terminals* $s, t \in V$ an *$s - t$ cut* is a cut $(S, V - S)$ with $s \in S$ and $t \in V - S$.
The problem of finding a minimum cost $s - t$ cut can be solved in polynomial time using a *Maximum Flow* algorithm.

The Multiway Cut problem:
Given a set of terminals $\{s_1, s_2, \ldots, s_k\} \subseteq V$ find a minimum cost set of edges in E whose removal disconnects all of the s_i from each other.
This problem is NP-Hard for $k \geq 3$.

An *isolating cut* for s_i is a set of edges whose removal disconnects s_i from all of the other terminals.

Multiway cut
1. For each $i = 1, \ldots k$ compute a minimum-weight isolating cut for S_i. Call it C_i.
2. Throw away the most expensive of the C_i. Output the union of the rest. Call it C.

Note: Finding a min-weight isolating cut for each s_i can be done in polynomial time. How?

Lemma Multiway cut is a $\left(2 - \frac{2}{k}\right)$-approximation algorithm.

Proof: Let A be a minimum cost multiway cut. The removal of A from E creates k components, each one containing exactly one of the s_i. Let A_i be the edges in A that connect the ith component to the rest of the graph. $A = \cup_{i=1}^{k} A_i$.
Note that

$$\sum_{i=1}^{k} w(A_i) = 2w(A).$$

A_i is an isolating cut for s_i so $w(C_i) \leq w(A_i)$ and

$$\begin{aligned} w(C) &\leq \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} w(C_i) \\ &\leq \left(1 - \frac{1}{k}\right) \sum_{i=1}^{k} w(A_i) = \left(2 - \frac{2}{k}\right) w(A). \end{aligned}$$