Lecture 3: Facility Location Problems

(Vazirani. Ch. 5)

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Let $G = (V, E)$ be a complete undirected graph with cost function $c()$ on its edges satisfying the triangle inequality.

The Minimum k-Center Problem: Let k be an integer. Find a set $S \subseteq V$, $|S| = k$ that minimizes

$$\max_{u \in V} \left\{ \min_{v \in S} c(u, v) \right\}.$$

Example: V is a set of cities and $c()$ the intercity transportation costs. The MKCP problem is to find locations for k warehouses that minimize the max distance from any city to a warehouse.

We will see a 2 approximation algorithm for MKCP.
We will also see that if $P \neq NP$ then, $\forall \epsilon > 0$ there is no $(2 - \epsilon)$ polynomial time approximation algorithm for solving it.

MKCP: Find S minimizing $\max_{u \in V} \{\min_{v \in S} c(u, v)\}$.

A *dominating set* of G is $S \subseteq V$ such that every vertex in $V - S$ is adjacent to a vertex in S.

Let $dom(G)$ be the size of a *smallest* dominating set in G. Finding $dom(G)$ is an NP-Hard problem.

Sort the edges of G so that

$$c(e_1) \le c(e_2) \le c(e_3) \cdots \le c_{e_{|E|}}.$$

Now set

$$G_i = (V, E_i), \quad E_i = \{e_1, e_2, e_3, \ldots, e_i\}.$$

Observation: The k center problem is equivalent to finding the smallest i such that $dom(G_i) \le k$.
If i^* is the smallest such index then $c(e_{i^*})$ is the cost of an optimal k-center.

MKCP: Find S minimizing $\max_{u \in V} \{\min_{v \in S} c(u, v)\}$.

A *dominating set* of G is $S \subseteq V$ such that every vertex in $V - S$ is adjacent to a vertex in S.
$dom(G)$ is the size of a *smallest* dominating set.

I is an *independent set* of graph $G = (V, E)$ if $I \subseteq V$ and $\forall u, v \in I$, $(u, v) \notin V$.

The *square* of G is $G^2 = (V, E')$ where

$(u, v) \in E' \Leftrightarrow (u, v) \in E$ or $\exists w$ s.t. $(u, w), (w, v) \in E$.

Lemma: Given Graph H, let I be any independent set in H^2. Then

$$|I| \le dom(H).$$

Proof: Let D be a minimum dominating set in H.
Every node u is a neighbor in H of some node $w \in D$ (or is in D). Suppose that u, v are both neighbors of $w \in D$ Then (u, v) is an edge in H^2 so u and v can not both be in I. Since only one neighbor in H of each $w \in D$ can be in I, $|I| \le |D|$ and the result follows.

MKCP: Find S minimizing $\max_{u \in V} \{\min_{v \in S} c(u, v)\}$.
$OPT = \max_{u \in V} \{\min_{v \in S} c(u, v)\}$.
$G_i = (V, E_i), \quad E_i = \{e_1, e_2, e_3, \ldots, e_i\}$.
If I is independent in H^2 then $|I| \le dom(H)$.

Minimum k-center
1. Construct $G_1^2, G_2^2, \ldots, G_{|E|}^2$.
2. Compute a *maximal* independent set M_i in each graph G_i^2.
3. Find smallest index i such that $|M_i| \le k$. Call this j.
4. Output M_j.

Lemma: $c(e_j) \le OPT$.
Proof: $\forall i < j$, $k < |M_i| \le dom(G_i)$.
Thus $i^* > i$ so $j \le i^*$ and $c(e_j) \le OPT$.

Lemma: MKC is a 2-approximation algorithm.

MKCP: Find S minimizing $\max_{u\in V}\{\min_{v\in S} c(u,v)\}$.
$OPT = \max_{u\in V}\{\min_{v\in S} c(u,v)\}$.
$G_i = (V, E_i), \quad E_i = \{e_1, e_2, e_3, \ldots, e_i\}$.
M_i is maximal independent set in G_i^2.
j is smallest index such that $|M_j| \le k$.
$c(e_j) \le OPT$.

Lemma: MKC is a 2-approximation algorithm.
Proof: First note that a maximal independent set I in a graph is also a dominating set. Otherwise, if vertex v is not dominated by I, then $I \cup \{v\}$ is a larger independent set. M_j therefore dominates G_j^2.

Now suppose that $u \in V$. Let $v \in M_j$ be the vertex that dominates it in G_j^2. There must then exist a w and two edges $e_i = (u, w)$ and $e_{i'} = (w, v)$ with $i, i' \le j$. By the triangle inequality

$$c(u,v) \le c(u,w) + c(w,v) \le 2C(e_j) \le 2OPT.$$

An Impossibility Result

A 2-approximation result is the 'best' that can be achieved for the k-center problem with triangle inequality.

Theorem: If $P \ne NP$ then there is no polynomial time approximation algorithm for the k-center problem with factor $2-\epsilon$, $\epsilon > 0$.

Proof: If a $(2-\epsilon)$-approximation algorithm $\mathcal{A}$ existed it would be able to solve the minimum dominating set problem in polynomial time.
Let $G = (V, E)$ and integer k be an instance of the dominating set problem. Construct complete graph $G' = (V, E')$ with edge costs

$$c(u,v) = \begin{cases} 1 & \text{if } (u,v) \in E \\ 2 & \text{if } (u,v) \notin E \end{cases}$$

If (i) $dom(G) \le k$ then G' has a k-center with cost 1.
If (ii) $dom(G) > k$ then the optimum cost of a k-center on G' is 2.

In case (i) $\mathcal{A}$ returns a cost of $\le 2-\epsilon$.
In case (ii) $\mathcal{A}$ returns a cost of ≥ 2.

Thus, to solve the dominating set problem we can build G', run the algorithm and check if the result ≥ 2 or not.

The Weighted Minimum k-Center Problem:
$G = (V, E)$ is a graph with cost function $c()$ that satisfies the triangle inequality. *There is also a weight function $w \to R^+$ on the vertices.*
Let $W \ge 0$.

The *Weighted Minimum k-Center Problem* is to find a set $S \subseteq V$, $w(S) \le W$ that minimizes

$$\max_{u\in V}\left\{\min_{v\in S} c(u,v)\right\}.$$

Let $wdom(G)$ be the weight of a minimum weight dominating set.

$$G_i = (V, E_i), \quad E_i = \{e_1, e_2, e_3, \ldots, e_i\}.$$

Observation: The weighted k center problem is equivalent to finding the smallest i such that $wdom(i) \le W$.
If i^* is the smallest such index then $c(e_{i^*})$ is the cost of an optimal k-center.

MKCP: Find S, $w(S) \le W$, minimizing $\max_{u\in V}\{\min_{v\in S} c(u,v)\}$.

For H be a vertex weighted graph let $s(u)$ be the *lightest* neighbor of u in H (where u is considered its own neighbor).

Let I be an independent set in H^2.

Lemma: Let $S = \{s(u) : u \in I\}$. Then

$$w(S) \le wdom(H).$$

Proof: Let D be a minimum weight dominating set of H. Every node u is a neighbor of some node $d \in D$ (where we consider u to be its own neighbor.)

Suppose that u, v are both neighbors of $d \in D$. Then (u, v) is an edge in H^2 so u and v can not both be in I. This means that for each $d \in D$ at most one neighbor in H of d can be in I. Let u be this neighbor if it exists. Then $w(s(u)) \le w(d)$. Summing gives:

$$\begin{aligned} w(S) &= \sum_{v\in S} w(v) \\ &\le \sum_{u\in I} w(s(u)) \\ &\le \sum_{d\in D} w(d) = wdom(H). \end{aligned}$$

$\boxed{\text{MKCP: Find } S,\ w(S) \le W, \text{ minimizing } \max_{u \in V} \{\min_{v \in S} c(u,v)\}.}$

$G_i = (V, E_i), \quad E_i = \{e_1, e_2, e_3, \ldots, e_i\}$.
$s_i(u)$ will denote the lightest neighbor of u in G_i.

Weighted Minimum k-center
1. Construct $G_1^2, G_2^2, \ldots, G_{|E|}^2$.
2. Compute a *maximal* independent set M_i in each graph G_i^2.
3. Compute $S_i = \{s_i(u) : u \in M_i\}$.
4. Find smallest index i such that $w(S_i) \le W$. Call this j.
5. Output S_j.

Lemma: $c(e_j) \le OPT$.

Lemma: WMKC is a 3-approximation algorithm.
Proof: First note that M_j is a dominating set of G_i^2. We can therefore cover V with *stars* of G_j^2 centered at vertices of M_j. By the triangle inequality the edges of these stars have cost (in G_j) of at most $2c(e_j)$. Then every $v \in V$ has some $u \in M_j$ such that $c(v,u) \le 2c(e_j)$.
For each star center $u_t \in M_j$ let s_t be the corresponding lightest neighbor, i.e., $s(u_t) = s_t$. Thus $S = \cup_t \{s_t\}$. Note that $c(u_t, s_t) \le c(e_j)$. (*Why?*)
This means that, $\forall v \in V$,

$$c(v, s_t) \le c(v, u_t) + c(u_t, s_t) \le 3c(e_j)$$

so WMKC is a 3 approximation solution.