Minimum Manhattan Network is NP-Complete

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Abstract

A *rectilinear path* between two points $p, q \in \mathbb{R}^2$ is a path connecting p and q with all its line segments horizontal or vertical segments. Furthermore, a *Manhattan path* between p and q is a rectilinear path with its length exactly $\mathrm{dist}(p, q) := |p.x - q.x| + |p.y - q.y|$, *i.e.*, the Manhattan distance between p and q.

Given a set T of n points in $\mathbb{R}^2$, a network G is said to be a *Manhattan network* on T, if for all $p, q \in T$ there exists a Manhattan path between p and q with all its line segments in G. For the given network G, let the length of G, denoted by $L(G)$, be the total length of all line segments of G. For the given point set T, the *Minimum Manhattan Network* (MMN) Problem is to find a Manhattan network G on T with the minimum $L(G)$.

Historical Review: Due to numerous applications in city planning, network layout, distributed algorithms, and VLSI circuit design, the MMN problem was firstly introduced in 1999 by J. Gudmundsson et al. In that paper, they highlighted three open problems: (1) whether or not MMN is NP-Hard; (2) whether or not PTAS exists for MMN; and (3) whether or not a 2-approximation algorithm exists for MMN.

In the past over ten years, much research was devoted to finding approximation algorithms for the MMN problem. Most combinatorial constructions rely on the decomposition of the input, by partitioning the input into several blocks (ortho-convex regions) that can be solved independently. J. Gudmundsson et al. proposed an $O(n^3)$-time 4-approximation algorithm, and an $O(n \log n)$-time 8-approximation algorithm. M. Benkert et al. proposed an $O(n \log n)$-time 3-approximation algorithm. They also described a mixed-integer programming (MIP) formulation of the MMN problem. After that, V. Chepoi et al. proposed a 2-approximation rounding algorithm by solving the linear programming relaxation of the MIP and also the notion of *Pareto Envelope* and a nice strip-staircase decomposition. In K. Nouioua's Ph.D thesis, a primal-dual based 2-approximation algorithm with running time of $O(n \log n)$ was presented. Later, Z. Guo et al. observed that the same approximation ratio can also be achieved in $O(n \log n)$ using a combinatorial construction. Despite the wealth of publications on the MMN problem, it is still open whether MMN is NP-Hard.

Our Results: In this paper, we shall prove that MMN is strongly NP-Complete, using the reduction from the well-known 3-SAT problem, which requires a number of gadgets. The gadgets have similar constructions, but play different roles in simulating the 3-SAT formula.

As a consequence of our reduction, it is easy to show that MMN is strongly NP-complete and there does not exist FPTAS algorithm for this problem under the assumption $P \neq NP$. This also answers the first problem of J. Gudmundsson et al., which was open for more than ten years.