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On the traveling salesman problem with a relaxed Monge matrix ☆

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Abstract

We show that the traveling salesman problem with a symmetric relaxed Monge matrix as distance matrix is pyramidally solvable and can thus be solved by dynamic programming. Furthermore, we present a polynomial time algorithm that decides whether there exists a renumbering of the cities such that the resulting distance matrix becomes a relaxed Monge matrix.

Keywords: Traveling salesman problem; Monge matrix; Pyramidally solvable case; Euclidean TSP; Algorithms

1. Introduction

The *traveling salesman problem* (TSP) is defined as follows. Given an $n \times n$ distance matrix $C = (c_{ij})$, find a cyclic permutation π of the set $\{1, 2, \ldots, n\}$ that minimizes the function

$$c(\pi) = \sum_{i=1}^{n} c_{i\pi(i)}$$

(the salesman must visit cities 1 to n in arbitrary order and wants to minimize the total travel length). This problem is known to be NP hard. For more information refer to Lawler et al. [11].

Several special cases of the TSP are solvable in polynomial time, due to special combinatorial structures of the distance matrix (see [9] and the recent survey [1]). Among them is the TSP with a Monge matrix.

An $n \times n$ matrix $C = (c_{ij})$ is called a *Monge matrix* if it satisfies the following conditions for all indices $i, j, k, l \in \{1, \ldots, n\}$ with $i < k$ and $j < l$:

$$c_{ij} + c_{kl} \leqslant c_{il} + c_{kj}. \tag{1}$$

Monge matrices are well known, due to their notorious role in combinatorial optimization (see the survey [2] and the references therein). Supnick [13] showed in 1957 that the TSP with a symmetric Monge matrix is solved by the tour $\langle 1, 3, 5, \ldots, 6, 4, 2\rangle$. (Here we use brackets $\langle .. \rangle$ to distinguish the cyclic representation of a permutation in the form $\pi = \langle 1, \pi(1), \pi(\pi(1)), \ldots\rangle$ from the alternative representation $\pi = (\pi(1), \pi(2), \ldots, \pi(n))$.)

In order to characterize optimal solutions of the TSP with asymmetric Monge matrices one needs the concept of *pyramidal tours*, i.e., permutations π with $\pi = \langle 1, i_1, i_2, \ldots, i_r, n, j_1, \ldots, j_{n-r-2}\rangle$, where $i_1 <$

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$i_2 < \cdots < i_r$ and $j_1 > \cdots > j_{n-r-2}$ hold. The TSP restricted to a class of matrices is called *pyramidally solvable* if for every matrix in this class there is an optimal tour that is pyramidal. Although the number of pyramidal tours on n cities is exponential in n, a minimum cost pyramidal tour can be determined in $O(n^2)$ time by a dynamic programming approach (cf. Gilmore et al. [9]).

It was shown by several authors (see, e.g., [9,1]) that the TSP restricted to asymmetric Monge matrices is pyramidally solvable, so it can be solved in $O(n^2)$ time.

Clearly, the combinatorial structure of a distance matrix depends on the numbering of the rows and columns. A matrix $C = (c_{ij})$ is called a *permuted Monge* matrix if there is a permutation σ of its rows and columns such that the permuted matrix $C_\sigma = (c_{\sigma(i)\sigma(j)})$ is a Monge matrix. A permuted Monge matrix can be recognized in $O(n^2)$ time [7]. So, the TSP on a permuted Monge matrix can also be solved in $O(n^2)$ time.

In this note, we introduce a relaxation of the Monge condition (1). An $n \times n$ matrix $C = (c_{ij})$ will be called a *relaxed Monge* matrix (RM-matrix, for short) if the following inequalities hold for all cities $i, i+1, j, l \in \{1, \ldots, n\}$ with $i+1 < j \neq l$:

$$c_{i,i+1} + c_{jl} \leqslant c_{il} + c_{j,i+1}, \tag{2}$$

$$c_{i+1,i} + c_{jl} \leqslant c_{i+1,l} + c_{ji}. \tag{3}$$

Note that diagonal elements of C are not involved in the definition of relaxed Monge matrices and thus may as well remain unspecified.

In Section 2 we show that the TSP restricted to symmetric RM-matrices is pyramidally solvable. We show that the system (2)–(3) is equivalent to the system of $O(n^2)$ inequalities, so the TSP with an RM-matrix can be recognized and solved in $O(n^2)$ time. Until now there was only one pyramidally solvable TSP case with the same property known, namely the TSP with a *Demidenko* matrix ([8], see also [9]).

In Section 3 we show that permuted RM-matrices can be recognized in polynomial time. What distinguishes the TSP restricted to symmetric RM-matrices from similar efficiently solvable special cases is that the following two properties hold simultaneously:

(1) It can be decided in polynomial time whether a matrix is a permuted RM-matrix.

(2) There does not exist a fixed optimal tour of the TSP restricted to an RM-matrix, i.e., a tour that depends only on n but not on the actual entries of the distance matrix.

All efficiently solvable cases known so far for which a property analogous to (1) holds have a fixed optimal tour (see [13,6] for permuted Supnick matrices and [10,4] for permuted Kalmanson matrices). This is different for the TSPs restricted to permuted symmetric RM-matrices: they can have differently structured optimal tours that can be found by applying a dynamic programming algorithm.

The *Euclidean TSP* is the TSP where the cities are represented by points in the two-dimensional Euclidean plane and the distances are measured according to the Euclidean metric. Given a specially structured distance matrix $C = (c_{ij})$, it is interesting to decide whether there exists a Euclidean point set with the distance matrix C. The combinatorial structure of Euclidean point sets that have the Monge property is rather primitive: if this set contains $n \geqslant 9$ points, they must lie on a common straight line (Quintas and Supnick [12]).

In Section 4 we characterize Euclidean point sets that fulfill the relaxed Monge property. Additional geometric properties of these matrices allow to reduce the complexity of the recognition algorithm for these matrices from $O(n^4)$ to $O(n^3)$.

2. The TSP on symmetric RM-matrices

We use a unified proof-technique for pyramidally solvable TSPs. This technique was essentially introduced by Van der Veen [15] and successfully applied in, e.g., Burkard and Van der Veen [3], Van der Veen [14] and Burkard et al. [1].

The idea is as follows. In order to prove that under certain conditions on the distance matrix there exists an optimal tour that is pyramidal, a *tour-improvement technique* (TI-technique) is used. Starting from an arbitrary tour τ, a sequence of tours $\tau_1, \tau_2, \ldots, \tau_s$ is constructed, with $\tau_1 = \tau$, such that

$$c(\tau_1) \geqslant c(\tau_2) \geqslant \cdots \geqslant c(\tau_s),$$

where $c(\tau_t)$ denotes the length of the tour τ_t and $s = s(\tau)$ is the smallest integer such that τ_s is a pyramidal tour. Note that if $s(\tau) < \infty$ for every tour τ,

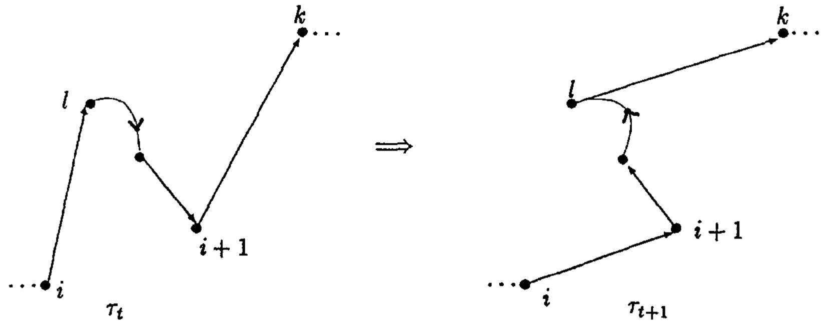


Fig. 1. Illustration to Step 2 of TI-algorithm (case $\tau_t(i) > i$).

then there is always an optimal tour that is pyramidal. Tour τ_{t+1} is obtained from τ_t by exchanging a number of arcs. This operation is called a *transformation.* A transformation is called *feasible* if the conditions on the distance matrix assure that the total length of the inserted arcs is no longer than the length of the removed arcs. In what follows, we use also the definition of a valley: An index $i \in \{1, \ldots, n\}$ is called a *valley* of a permutation τ if $i < \min\{\tau^{-1}(i), \tau(i)\}$.

Theorem 2.1. *The TSP restricted to symmetric RM-matrices is pyramidally solvable.*

Proof. Consider the following TI-algorithm.

TI-technique
Input: A tour τ.
Output: A pyramidal tour.
Step 0: Set $\tau_1 := \tau$ and $t := 1$.
Step 1: Find the smallest valley $i+1$ that is greater than 1.
If τ_t does not contain such a valley, then STOP: τ_t is a pyramidal tour.
Step 2: Transformation:
if $\tau_t(i) > i$ **then**
Let $\tau_t(i) = l$ and $\tau_t(i+1) = k$.
Obtain τ_{t+1} from τ_t by replacing arcs $[i, l]$ and $[i+1, k]$ by $[i, i+1]$, $[l, k]$ and reversing the subpath $[l, \tau_t(l), \ldots, i+1]$.
Return to Step 1 with $t := t+1$.
else
Let $\tau_t^{-1}(i) = l$ and $\tau_t^{-1}(i+1) = k$.
Obtain τ_{t+1} from τ_t by replacing arcs $[l, i]$ and $[k, i+1]$ by $[i+1, i]$, $[k, l]$ and reversing the subpath $[i+1, \tau_t(i+1), \ldots, l]$.
Return to Step 1 with $t := t+1$.

The feasibility of the transformation in Step 2 follows immediately from inequalities (2)–(3). Since the smallest valley (greater than 1) is increased in every step, the algorithm ends after a finite number of steps. Starting with an arbitrary optimal tour, the algorithm constructs an optimal tour, which is pyramidal. □

Proposition 2.2. *A symmetric RM-matrix can be recognized in* $O(n^2)$ *time.*

Proof. We claim that in the symmetric case the system (2)–(3) is equivalent to the system

$$c_{i,i+1} + c_{i+2,l} \leqslant c_{il} + c_{i+2,i+1}, \tag{4}$$

$$c_{i,i+1} + c_{l,i+2} \leqslant c_{i,i+2} + c_{l,i+1}, \tag{5}$$

with $i, i+1, i+2, l \in \{1, \ldots, n\}$ and $i+2 < l$. The proposition follows immediately from this claim.

It is clear that (4)–(5) is a subsystem of (2)–(3). To prove that (4)–(5) imply (2)–(3), consider the differences

$$\Delta(i, i+1, j, l) = c_{i,i+1} + c_{jl} - c_{il} - c_{j,i+1}$$

with $i, i+1, j, l \in \{1, \ldots, n\}$ and $i+1 < j \neq l$. It follows from (4)–(5) that $\Delta(i, i+1, j, l) \leqslant 0$ if $\min\{j, l\} = i+2$. If $\min\{j, l\} > i+2$, then the following recursion holds:

$$\Delta(i, i+1, j, l) = \Delta(i, i+1, i+2, l) + \Delta(i+1, i+2, l, j).$$

(Here we used the fact that C is a symmetric matrix.) This recursion together with (4)–(5) guarantees the inequalities $\Delta(i, i+1, j, l) \leqslant 0$ for all $i, i+1, j, l \in \{1, \ldots, n\}$ with $i+1 < j \neq l$. This proves the proposition. □

As an open problem we pose the question to decide the computational complexity of the TSP with an asymmetric RM-matrix. This problem is not pyramidally solvable, as the example below shows. It is easy to see that the matrix

$$C = \begin{pmatrix} * & 0 & 1 & 1 & 1 & 1 \\ 0 & * & 0 & 1 & 1 & 0 \\ 1 & 1 & * & 1 & 0 & 1 \\ 0 & 1 & 0 & * & 0 & 0 \\ 1 & 1 & 1 & 0 & * & 0 \\ 1 & 1 & 0 & 1 & 0 & * \end{pmatrix}$$

is an RM-matrix and the TSP with C has a unique optimal tour $\tau = \langle 1, 2, 6, 3, 5, 4 \rangle$.

3. Recognizing permuted RM-matrices

In this section, we consider the following problem:

Given an $n \times n$ distance matrix $C = (c_{ij})$, does there exist a renumbering of the cities, i.e., a permutation σ of the rows and columns of C, such that the resulting matrix $C_\sigma = (c_{\sigma(i)\sigma(j)})$ is an RM-matrix?

If such a permutation σ exists, then the matrix C is called a *permuted RM-matrix* and σ is called an *RM-permutation.*

Note that in this section we do not require that C be symmetric. Though we only succeeded in proving the polynomial solvability of the TSP restricted to a symmetric RM-matrix, we still hope that the asymmetric case can be solved in polynomial time as well.

A permuted RM-matrix can be recognized by an algorithm that is based on the next lemma. Suppose that $u-1$ cities have already been chosen and placed in an RM-permutation at the places $1, 2, \ldots, u-1$. Let these cities be renumbered as $1, 2, \ldots, u-1$. Then the city v that can be assigned to place u should be chosen from the conditions (2)–(3). We can express this by $c_{u-1,v} + c_{kl} \leqslant c_{u-1,l} + c_{kv}$ and $c_{v,u-1} + c_{kl} \leqslant c_{vl} + c_{k,u-1}$ with k, l: $k \neq v$, $l \neq v$ and $k \neq l > u-1$. If several candidates for v fulfill these conditions, then any of them can be assigned to u, as the following lemma shows.

Lemma 3.1. *If for a given $n \times n$ RM-matrix there are two indices u and v: $1 < u < v \leqslant n$ such that*

$$c_{u-1,v} + c_{kl} \leqslant c_{u-1,l} + c_{kv}, \tag{6}$$

$$c_{v,u-1} + c_{kl} \leqslant c_{vl} + c_{k,u-1}, \tag{7}$$

for all k, l: $k \neq v$, $l \neq v$ and $k \neq l > u-1$, then the permutation

$$\mu = (1, 2, \ldots, u-1, v, u+1, \ldots, v-1, u, v+1, \ldots, n)$$

is an RM-permutation.

Proof. First, we claim that the following equalities

$$c_{uk} - c_{vk} = c_{u,u-1} - c_{v,u-1}, \tag{8}$$

$$c_{kv} - c_{ku} = c_{u-1,v} - c_{u-1,u} \tag{9}$$

hold for all $k > u$, $k \neq v$. Indeed, it follows from the definition of an RM-matrix that

$$c_{u-1,u} + c_{kv} \leqslant c_{u-1,v} + c_{ku},$$

$$c_{u,u-1} + c_{vk} \leqslant c_{uk} + c_{v,u-1},$$

for all $k > u$. On the other hand, system (6)–(7) contains the inequalities

$$c_{u-1,v} + c_{ku} \leqslant c_{u-1,u} + c_{kv},$$

$$c_{v,u-1} + c_{uk} \leqslant c_{vk} + c_{u,u-1}$$

with $k > u$. Combining these two systems gives us the equalities (8)–(9).

Now we prove that μ is an RM-permutation, i.e., the inequalities

$$c_{\mu(i),\mu(i+1)} + c_{\mu(j),\mu(k)} \leqslant c_{\mu(i),\mu(k)} + c_{\mu(j),\mu(i+1)} \tag{10}$$

and

$$c_{\mu(i+1),\mu(i)} + c_{\mu(j),\mu(k)} \leqslant c_{\mu(i+1),\mu(k)} + c_{\mu(j),\mu(i)} \tag{11}$$

hold for all $i, i+1, j, k \in \{1, 2, \ldots, n\}$ with $i+1 < j \neq k$. We restrict ourselves to considering only inequalities (10) and distinguish the following cases:

- $i < u-1$, or $i > v$, or $u < i < v-1$ and $j \neq v$, $k \neq v$;
- $i = u-1$;
- $i = u$;
- $u < i < v-1$ and ($j = v$ or $k = v$);
- $i = v-1$;
- $i = v$.

Clearly, in the first case (10) is equivalent to (2), which follows from the fact that C is an RM-matrix.

If $i = u-1$, then $\mu(i) = u-1$, $\mu(i+1) = v$, and (10) is equivalent to (6).

If $i = u$ and $u+1, j, k \neq v$, then (10) has the form

$$c_{v,u+1} + c_{jk} \leqslant c_{vk} + c_{j,u+1}.$$

Taking into account (8), the last inequality can be rewritten as

$$c_{u,u+1} - (c_{vk} + c_{u,u-1} - c_{v,u-1}) + c_{jk} \leqslant c_{uk} - (c_{vk} + c_{u,u-1} - c_{v,u-1}) + c_{j,u+1},$$

or

$$c_{u,u+1} + c_{jk} \leqslant c_{uk} + c_{j,u+1},$$

which follows from the definition of an RM-matrix.

If $i = u$ and $u+1 = v$, then $\mu(i) = u+1$, $\mu(i+1) = u$, and (10) follows again from the fact that $C = (c_{ij})$ is an RM-matrix.

If $i = u$, $u+1 < v$ and $j = v$, then we have the inequality

$$c_{v,u+1} + c_{uk} \leqslant c_{vk} + c_{u,u+1},$$

which is transformed using (8)–(9) into the inequality

$$c_{u,u+1} + c_{vk} \leqslant c_{uk} + c_{v,u+1}.$$

The case $i = u$, $u+1 < v$ and $k = v$ is similar to the previous one.

If $u < i < v-1$ and $j = v$ or $k = v$, we can use again (8)–(9) and transform the inequality into one that follows from the definition of an RM-matrix.

If $i = v$ or $i = v-1 = u$, then again (10) follows directly from the definition of an RM-matrix. If $i = v-1$ and $v-1 \neq u$, then $v-1 > u$ and, using (9), we transform (10) into an inequality that follows immediately from the definition of an RM-matrix. Thus the lemma is proved. □

Due to the remarks at the beginning of this section, we get therefore the following result:

Theorem 3.2. *It can be decided in* $O(n^4)$ *time whether* $n \times n$ *matrix* $C = (c_{ij})$ *is a permuted RM-matrix. If it is, permutation* σ *is explicitly determined within this time bound.*

Proof. First, try all n cities as candidates for the first place. Having chosen the first city, transform matrix $C = (c_{ij})$ into $C' = (c'_{ij})$ by subtracting constants from rows and columns: $c'_{ij} = c_{ij} - c_{1j} - c_{i1}$, $i, j \in \{1, 2, \ldots, n\}$. (We suppose here that $c_{11} = 0$.)

Suppose that the second city has been chosen. It follows from (3) with $i = 1$ that $c'_{2l} \geqslant c'_{jl}$ for $l, j \in \{3, \ldots, n\}$, $i \neq j$. This means that a candidate for the second city (if there exist any) can be found by comparing the rows in C' in $O(n^2)$ time. If there is more than one candidate, any of them can be chosen, as the lemma shows.

Having chosen the second city, transform matrix $C = (c_{ij})$ into $C' = (c'_{ij})$ with $c'_{ij} = c_{ij} - c_{2j} - c_{i2}$, $i, j \in \{2, \ldots, n\}$ and find the third city, and so on.

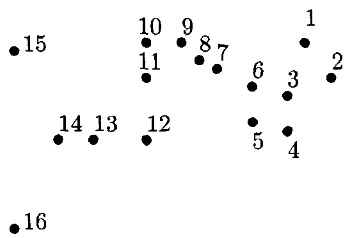
So, the algorithm takes $O(n^4)$ operations overall. □

4. The Euclidean TSP with specially structured matrices

This section deals with planar Euclidean point sets whose distance matrices are RM-matrices. We use here an approach presented in [5]. Let $S := (v_1, v_2, \ldots, v_n) \subseteq \mathbb{R}^2$ be a sequence of n points in the Euclidean plane and let $C = (c_{ij})$ denote its distance matrix defined by $c_{ij} = d(v_i, v_j)$, where $d(x, y)$ denotes the Euclidean distance between points x and y.

A sequence S of points is called an *RM-sequence* if the corresponding distance matrix C is an RM-matrix.

For $x, y, z \in \mathbb{R}^2$, denote by $h(x, y, z) = \{p \in \mathbb{R}^2 \mid d(x, p) - d(y, p) = d(x, z) - d(y, z)\}$ the set of points $p \in \mathbb{R}^2$ that lie on one (uniquely determined) branch of the hyperbola with focal points at x and y. Furthermore let $H(x, y, z) = \{p \in \mathbb{R}^2 \mid d(x, p) - d(y, p) \geqslant d(x, z) - d(y, z)\}$ denote the set of points



1	(34,31)	9	(20,31)
2	(37,27)	10	(16,31)
3	(32,25)	11	(16,27)
4	(32,21)	12	(16,20)
5	(28,22)	13	(10,20)
6	(28,26)	14	(6,20)
7	(24,28)	15	(1,30)
8	(22,29)	16	(1,10)

Fig. 2. An RM-sequence of points with coordinates.

$p \in \mathbb{R}^2$ in the infinite region bounded by $h(x, y, z)$ that does not contain the focal point x.

Proposition 4.1. *A point sequence $S = (v_1, \ldots, v_n)$ is an RM-sequence iff for each p, $4 \leqslant p \leqslant n$, point v_p lies within the region*

$$H_p = H_{p-1} \cap H(v_{p-3}, v_{p-1}, v_{p-2}) \cap H(v_{p-2}, v_{p-1}, v_{p-3}),$$

where $H_3 = \mathbb{R}^2$.

Proof. The proof, based on inequalities (4)–(5), is similar to the proof of Theorem 3.1 of [5]. □

Proposition 4.1 allows to generate RM-sequences of points in the Euclidean plane. Fig. 2 gives an illustration of an RM-sequence of 16 points. The optimal (pyramidal) tour is $\langle 1, 2, 3, 4, 5, 12, 13, 14, 16, 15, 11, 10, 9, 8, 7, 6 \rangle$.

Geometric properties of RM-sequences allow to reduce the complexity of the recognition algorithm for Euclidean permuted RM-matrices from $O(n^4)$ to $O(n^3)$. We claim that in this special case the number of pairs that can be assigned to the first and second places is bounded by a constant. This implies directly the stated complexity result.

Indeed, consider a Euclidean RM-matrix and suppose that there are two indices k_1 and l_1 ($k_1 < l_1$) such that

$$c_{k_1 l_1} + c_{ij} \leqslant c_{k_1 j} + c_{i l_1}, \quad (12)$$

$$c_{l_1 k_1} + c_{ij} \leqslant c_{l_1 j} + c_{i k_1} \quad (13)$$

for all $i, j \in \{1, \ldots, n\} \setminus \{k_1, l_1\}$. Such a pair $\{k_1, l_1\}$ is a candidate for the first and second places in an RM-permutation.

If $k_1 > 1$, then the definition of RM-matrices yields

$$c_{k_1-1,k_1} + c_{l_1 j} \leqslant c_{k_1-1,j} + c_{l_1 k_1}.$$

Using (12)–(13), we get

$$c_{l_1 k_1} + c_{k_1-1,j} \leqslant c_{l_1 j} + c_{k_1-1,k_1}$$

and, therefore,

$$c_{l_1 k_1} + c_{k_1-1,j} = c_{l_1 j} + c_{k_1-1,k_1}$$

for $j \in \{k_1, k_1+1, \ldots, n\} \setminus \{l_1\}$. This means that all points $j \in \{k_1, k_1+1, \ldots, n\} \setminus \{l_1\}$ lie on a branch of the hyperbola with focal points at $k_1 - 1$ and l_1.

If $k_1 = 1$, then all points $j \in \{1, 3, 4, \ldots, n\} \setminus \{l_1\}$ lie on a branch of the hyperbola with focal points at 2 and l_1.

Suppose now that there is another pair of points $\{k_2, l_2\}$ ($k_1 \leqslant k_2 < l_2$) with the same property. Since two branches of hyperbolas with different focal points contain no more than four common points we get $k_2 \geqslant n - 6$ and therefore only a constant number of possibilities for k_2 and l_2. Thus the claim is proved.

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