Four Point Conditions and Exponential Neighborhoods for Symmetric TSP

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Abstract

In most of the known polynomially solvable cases of the symmetric travelling salesman problem (TSP) which result from restrictions on the underlying distance matrices, the restrictions have the form of so-called four-point conditions (the inequalities involve four cities). In this paper we treat *all possible* (symmetric) four-point conditions and investigate whether the corresponding TSP can be solved in polynomial time. As a by-product of our classification we obtain new *families* of exponential neighborhoods for the TSP which can be searched in polynomial time and for which conditions on the distance matrix can be formulated so that the search for an optimal TSP solution can be restricted to these exponential neighborhoods.

1 Introduction.

The travelling salesman problem (TSP) is a well known problem of combinatorial optimization. In the symmetric TSP, given a symmetric $n \times n$ distance matrix $C = (c_{ij})$, one looks for a cyclic permutation τ of the set $\{1, 2, \dots, n\}$ that minimizes the function $c(\tau) = \sum_{i=1}^{n} c_{i\tau(i)}$. The value $c(\tau)$ is called the *length* of the permutation τ. We will in the following refer to the items in τ as points or cities.

The TSP is an NP-hard problem [10]. There exist, however, special cases of the TSP which can be solved in polynomial time. For a survey on so-called *efficiently solvable cases* of the TSP see [4, 11, 15]. Many of the well known efficiently solvable cases of the TSP result from imposing special conditions on the underlying distance matrix. A large subclass of such conditions is formed by the so-called *four-point* conditions. Let i, j, k and l be four points with $1 \le i < j < k < l \le n$. A symmetric distance matrix for these points contains six different entries which correspond to the six edges connecting these points. It is possible to form three sets of pairs of non-incident edges: $\{(i, j), (k, l)\}$, $\{(i, k), (j, l)\}$, and $\{(i, l), (j, k)\}$. We denote the lengths of these pairs as $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, correspondingly:

$$\mathcal{A} = c_{ij} + c_{kl},\ \mathcal{B} = c_{ik} + c_{jl},\ \mathcal{C} = c_{il} + c_{jk}.$$

A *four-point* condition defines relationships among the values $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$, which have to be satisfied for all possible choices of the indices i,j,k and l with $1 \le i < j < k < l \le n$.

The typical approach to show that the TSP can be solved in polynomial time for matrices which fulfill a certain four-point condition works as follows. First, it is shown that there exists an optimal TSP tour within a set $\mathcal{S}$ of specially structured permutations. Second, it is shown that an optimal permutation can be found within this subset of permutations $\mathcal{S}$.

The technique which is usually used to show that an optimal TSP tour can be found in a special set, is the so-called *tour-improvement technique*. The idea is as follows. Starting from an arbitrary tour τ, a sequence of tours $\tau_1, \tau_2, \dots, \tau_T$ is constructed, with $\tau_1 = \tau$, such that

$$c(\tau_1) \ge c(\tau_2) \ge \cdots \ge c(\tau_T),$$

where τ_T is a tour in the specially structured set. Four-point conditions are used to prove the inequalities $c(\tau_t) \ge c(\tau_{t+1})$, $(t = 1, \dots, T)$. In some lucky cases, the special subset of permutations contains only one tour and thus no algorithm for finding an optimal tour is needed in this case. As an example for such a case, consider the Supnick TSP [25], i.e., the TSP restricted to matrices that satisfy the conditions $\mathcal{A} \le \mathcal{B}$, $\mathcal{B} \le \mathcal{C}$. An optimal tour for the Supnik TSP is given by $\sigma_{\mathrm{Smin}} = \langle 1, 3, 5, 7, 9, \dots, 8, 6, 4, 2, 1 \rangle$.

Another well-known solvable case of the TSP is the TSP restricted to Kalmanson matrices [16], i.e., to matrices that satisfy the conditions $\mathcal{A} \le \mathcal{B}$, $\mathcal{B} \ge \mathcal{C}$. An optimal tour is given by $\tau_{\mathrm{Kmin}} = \langle 1, 2, 3, \dots, n-1, n, 1 \rangle$.

Note that both Supnick and Kalmanson matrices satisfy the condition $\mathcal{A} \le \mathcal{B}$. The TSP with a distance matrix satisfying this condition is known as the Demidenko TSP [9]. The Demidenko TSP can be solved in $O(n^2)$ time ([9], see also [4, 11]) by finding an optimal pyramidal tour. A tour $\tau = \langle 1, i_1, i_2, \dots, i_r, n, j_1, j_2, \dots, j_{n-r-2}, 1 \rangle$ is called a *pyramidal* tour if $i_1 < i_2 < \cdots < i_r$ and $j_1 > j_2 > \cdots > j_{n-r-2}$. The TSP restricted to a class of matrices is called *pyramidally solvable* if for every matrix in this class there is an optimal tour that is pyramidal.

The set of pyramidal tours is one of the first well studied exponential neighborhoods that can be searched

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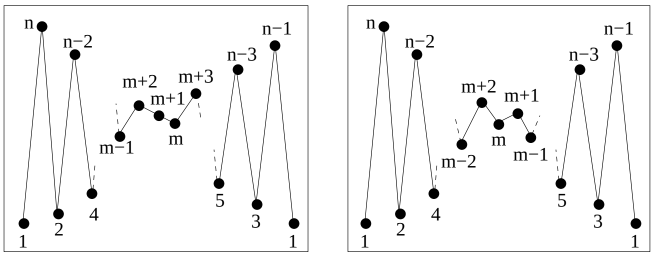


(a) (b)

Figure 1: Specially structured tours: (a) σ_{Smax} – optimal tour for Supnick MaxTSP with $n = 2m+1$ and m odd; (b) σ_{Smax1} – optimal tour for Supnick MaxTSP with $n = 2m$ and m odd.

	$\mathcal{A} \le \mathcal{B}$	$\mathcal{A} \ge \mathcal{B}$	$\mathcal{A} \le \mathcal{C}$	$\mathcal{A} \ge \mathcal{C}$	$\mathcal{B} \le \mathcal{C}$	$\mathcal{B} \ge \mathcal{C}$
$\mathcal{A} \le \mathcal{B}$:	$O(n^2)$ [9]	$O(1)$ [11,28]	$O(n^2)$ [9,27]	$O(1)$ [16]	$O(1)$ [25]	$O(1)$ [16]
$\mathcal{A} \ge \mathcal{B}$:		NPhard [8]	$O(1)$ [Sec 2.2]	NPhard [8,26]	$O(n)$ [16,22,23]	$O(n)$ [25]
$\mathcal{A} \le \mathcal{C}$:			$O(n^2)$ [27]	$O(1)$ [11,28]	$O(1)$ [Sec 2.3]	$O(1)$ [Sec 2.3]
$\mathcal{A} \ge \mathcal{C}$:				NPhard [26]	$O(1)$ [Sec 2.4]	$O(1)$ [Sec 2.4]
$\mathcal{B} \le \mathcal{C}$:					$O(?)$ [Sec 5]	$O(n^2)$ [Sec 3]
$\mathcal{B} \ge \mathcal{C}$:						$O(n^4)$ [Sec 4]

Table 1: Classification of Four Point Conditions

in polynomial time. Studies of other exponential neighborhoods have been extensively reported in the literature, not only for the TSP, but also for other combinatorial optimization problems (see [3, 6, 13, 21], and the surveys [1, 7]). To the best of our knowledge, until recently the pyramidal tours neighborhood remained, however, the only exponential neighborhood for the symmetric TSP for which classes of matrices were known such that the search for an optimal solution of the resulting special cases of the TSP can be restricted to the considered exponential neighborhood.

Outline of the extended abstract. In the next section we will classify all possible four-point conditions and summarize results related to the corresponding TSPs. In Sections 3 we describe a new polynomially solvable case of the TSP that can be solved in $O(n^2)$ time. In Sections 4 and 5 we deal with the TSP with relaxed Kalmanson and relaxed Supnick matrices, respectively. As a result, we come up with *families* of new exponential neighborhoods which can be searched in polynomial time and for which conditions on the distance matrix can be formulated so that the search for an optimal TSP solution can be restricted to these exponential neighborhoods.

Notations. The set of all permutations of $\{1, 2, \ldots, n\}$ will be denoted as S_n. For $\tau \in S_n$, we denote by τ^{-1} the inversion of τ, i.e., the permutation for which $\tau^{-1}(i)$ is the predecessor of i in the permutation τ, for $i = 1, \ldots, n$. For $k > 1$, we define $\tau^k(i)$ as $\tau(\tau^{(k-1)}(i))$ and $\tau^{-k}(i)$ as $\tau^{-1}(\tau^{-(k-1)}(i))$. In what follows we use also a cyclic representation of a cyclic permutation in the form $\tau = \langle i, \tau(i), \tau^2(i), \ldots, \tau^{-2}(i), \tau^{-1}(i), i\rangle$, and we refer to it as a tour. A pair (i, j) with $j = \tau(i)$ is referred as an *arc* of the tour τ. For the symmetric TSP, tour τ has the same length as τ^{-1}. The orientation of the arcs is, however, important for some operations on the tours. If the orientation of an arc is not known (or not important), we will use the term *edge*.

In what follows we will often refer to peaks and valleys of a tour. An index $i \in \{1, \ldots, n\}$ is a *peak* of a permutation τ if $i > \max\{\tau^{-1}(i), \tau(i)\}$ and a valley if $i < \min\{\tau^{-1}(i), \tau(i)\}$. An index which is neither a peak nor valley is called *intermediate*.

In this paper we will also deal with the maximization version of the TSP, where the objective is to maximize the function $c(\tau)$. This problem is called the *MaxTSP*. Although the MaxTSP reduces to the TSP (and vice versa) by replacing the matrix (c_{st}) by the matrix $(-c_{st})$, the special combinatorial structure that leads to a polynomially solvable case for the TSP does not necessarily yield a polynomially solvable case for the MaxTSP. For example, the MaxTSP restricted to Demidenko matrices is NP-hard [8]. To be consistent with the terminology used in previously published papers, we will use both, the TSP and its MaxTSP equivalent.

2 Four-point conditions: classification.

There are six possibilities to use the binary relations "$\le$" and "$\ge$" to define pairwise relationships for the sums $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$. As was mentioned above, for some of the four-point conditions the corresponding TSP is NP-hard. Therefore we also consider all possible combinations of pairs of binary relations. The results of our classification are summarized in Table 1.

2.1 $\mathcal{A} \le \mathcal{B}$. The TSP with a distance matrix satisfying this condition is the Demidenko TSP, which is pyramidally solvable [9]. Adding additional conditions will lead in some cases to straightforward solutions.

$\mathcal{A} \le \mathcal{B}$, $\mathcal{A} \ge \mathcal{B}$. These conditions yield $c_{ij} + c_{kl} = c_{ik} + c_{jl}$, for all i,j,k and l, with $i < j < k < l$. It can be shown that in this case matrix (c_{st}) is a sum matrix, i.e. $c_{st} = u_s + v_t$, for some vectors u and v. So, the corresponding TSP is trivial, with all tours having the same length [11, 28].

$\mathcal{A} \le \mathcal{B}$, $\mathcal{A} \le \mathcal{C}$. It is an open question whether it is possible to use these conditions to specify a subset of the set of pyramidal tours such that an optimal tour can be found faster than in $O(n^2)$ time.

$\mathcal{A} \le \mathcal{B}$, $\mathcal{A} \ge \mathcal{C}$. It can easily be shown that in this case the matrix belongs to a special subset of Kalmanson matrices [16], with τ_{Kmin} (see the Introduction) being an optimal tour.

$\mathcal{A} \le \mathcal{B}$, $\mathcal{B} \le \mathcal{C}$. This is the Supnick TSP [25], with σ_{Smin} being an optimal tour.

$\mathcal{A} \le \mathcal{B}$, $\mathcal{B} \ge \mathcal{C}$. This is the well known Kalmanson TSP [16], with τ_{Kmin} being an optimal tour.

2.2 $\mathcal{A} \ge \mathcal{B}$. This is the Demidenko MaxTSP, which is NP-hard [8]. Adding additional conditions will lead in some cases to polynomially solvable cases.

$\mathcal{A} \ge \mathcal{B}$, $\mathcal{A} \le \mathcal{C}$. It can be shown that σ_{Smin} is an optimal tour for the corresponding TSP.

$\mathcal{A} \ge \mathcal{B}$, $\mathcal{A} \ge \mathcal{C}$. It follows from [8] and [26] that the TSP remains NP-hard.

$\mathcal{A} \ge \mathcal{B}$, $\mathcal{B} \le \mathcal{C}$. This is the well-known Kalmanson MaxTSP [16, 22, 23]. Kalmanson has shown that for n odd, $n = 2m + 1$, the optimum TSP tour is $\tau_{\mathrm{K}} = \langle 1, m+2, 2, m+3, 3, m+4, \ldots, m, n, m+1, 1\rangle$. If n is even, then an optimum TSP tour can be found among m specially structured tours $\tau_{\mathrm{K}0}, \tau_{\mathrm{K}1}, \ldots, \tau_{\mathrm{K}i}, \ldots, \tau_{\mathrm{K}m-1}$ (see [16, 8] for a formal definition of the tours).

$\mathcal{A} \ge \mathcal{B}$, $\mathcal{B} \ge \mathcal{C}$. This is the well-known Supnick MaxTSP [25, 24, 19], which is solved by the tour $\sigma_{\mathrm{Smax}} = \langle 1, n, 2, n-2, \ldots, m-1, m+2, m+1, m, m+3, \ldots, 5, n-3, 3, n-1, 1\rangle$ for n odd, $n = 2m+1$ and $\sigma_{\mathrm{Smax1}} = \langle 1, n, 2, n-2, \ldots, m-2, m+2, m, m+1, m-1, \ldots, 5, n-3, 3, n-1, 1\rangle$ for n even, $n = 2m$ (see Figure 1(a),(b) for an illustration for odd m).

2.3 $\mathcal{A} \le \mathcal{C}$. We will refer to the corresponding matrices as Van der Veen matrices. If we consider the inequalities defined by $\mathcal{A} \le \mathcal{C}$ only for indices i, j, k, l with $k = j + 1$, then the corresponding TSP, as it was shown by Van der Veen [27], is pyramidally solvable. It seems that adding additional constraints, i.e. considering indices k, j with $k > j + 1$, does not simplify the problem. Adding additional four-point conditions will lead to straightforward solutions:

$\mathcal{A} \le \mathcal{C}$, $\mathcal{A} \ge \mathcal{C}$. The corresponding matrix (c_{ij}) is a sum matrix.

$\mathcal{A} \le \mathcal{C}$, $\mathcal{B} \le \mathcal{C}$. It can be shown that σ_{Smin} is an optimal tour for the corresponding TSP.

$\mathcal{A} \le \mathcal{C}$, $\mathcal{B} \ge \mathcal{C}$. Clearly, $\mathcal{B} \ge \mathcal{A}$, and therefore the class of the TSPs restricted to such matrices is a special subclass of the Kalmanson TSP, with τ_{Kmin} being an optimal tour.

2.4 $\mathcal{A} \ge \mathcal{C}$. The corresponding TSP remains NP-hard, see [26].

$\mathcal{A} \ge \mathcal{C}$, $\mathcal{B} \le \mathcal{C}$. It can be shown that the corresponding TSP is a special case of the Kalmanson MaxTSP. For n odd, $n = 2m+1$, the optimum TSP tour is $\tau_{\mathrm{K}} = \langle 1, m+2, 2, m+3, 3, m+4, \ldots, m, n, m+1, 1\rangle$. It follows from Theorem 4.1 in [8] and condition $\mathcal{A} \ge \mathcal{C}$, that for n even, $n = 2m$, the set of tour-candidates for an optimal tour is reduced to the unique tour $\tau_{\mathrm{K}0} = \langle 1, m+1, 2, m+3, 4, m+5, 6, \ldots, 7, m+6, 5, m+4, 3, m+2, 1\rangle$.

$\mathcal{A} \ge \mathcal{C}$, $\mathcal{B} \ge \mathcal{C}$. This is a special case of the MaxTSP restricted to Van der Veen matrices. In Section 4 (see Corollary 4.3) we will show that, for n even, $n = 2m$, the optimal tour can be found by taking the longer tour of the following two specially structured tours, tour $\sigma_{\mathrm{S}max1}$ which is an optimal tour for the Supnick MaxTSP (see Figure 1(b)) and tour $\sigma_{\mathrm{S}max2}$. For n odd, the optimal tour can be found as maximum among three specially structured tours, σ_{Smax} (shown on Figure 1(a)), and $\sigma_{\mathrm{S}max3}$, and $\sigma_{\mathrm{S}max4}$ (cf. the proof of Corollary 4.3).

2.5 $\mathcal{B} \le \mathcal{C}$. The status of the TSP restricted to this set of matrices remains open. It is possible, however, to identify polynomially solvable sub-cases when, as it is shown in Section 5, an optimal tour can be found in a special exponential neighborhood.

$\mathcal{B} \le \mathcal{C}$, $\mathcal{B} \ge \mathcal{C}$. The corresponding TSP reduces to the well-known Lawler TSP [17], and can therefore be solved in $O(n^3)$ time by reduction to the assignment problem. In Section 3 we show, however, that the problem can be solved even in $O(n^2)$ time.

2.6 $\mathcal{B} \ge \mathcal{C}$. This is a new case to which we will refer as the TSP with a Relaxed Kalmanson matrix. We show in Section 4 that an optimal tour can be found in $O(n^4)$ time in the set of tours, which is a proper subset of the well known twisted permutations [2].

3 Special Case of Lawler TSP.

Conditions $\mathcal{B} \le \mathcal{C}$ and $\mathcal{B} \ge \mathcal{C}$ yield $\mathcal{B} = \mathcal{C}$. We show first that the corresponding TSP is equivalent to a special case of the well-known Lawler TSP [17] with an upper-diagonal distance matrix. To show this, we transform matrix (c_{ij}) into the matrix $C' = (c'_{ij})$ by subtracting constants from all rows and columns: $c'_{ij} = c_{ij} - c_{1j} - c_{i1}$, (we assume here that $c_{11} = 0$). We claim that if matrix (c_{ij}) satisfies conditions $\mathcal{B} = \mathcal{C}$, then $c'_{ij} = a_j$ for all $j > i$, where $a_1 (= 0), a_2, \ldots, a_{n-1}$ are some constants. Indeed, $c'_{1j} = 0$ by construction. It follows from $\mathcal{B} = \mathcal{C}$, that $c'_{1j} + c'_{2k} = c'_{1k} + c'_{2j}$, for all $k > j > 2$, i.e. $c'_{2k} = c'_{2j} (= a_2)$. Having proved that $c'_{2j} = a_2$, we immediately get $c'_{3j} = a_3$ by the same arguments, and so on. This proves the claim.

Matrix (c'_{ij}) can be transformed into an *upper diagonal matrix* (d_{ij}), with $d_{ij} = 0$ for $i > j$, by subtracting constant a_j from column j for $j = 2, 3, \ldots, n-1$. It was discovered by Lawler [17] that the TSP with an upper diagonal matrix can be solved in $O(n^3)$ time, by reduction of the TSP to the assignment problem. This complexity can be improved as shown in the following

Proposition 3.1. *The TSP with the symmetric distance matrix (c_{ij}) with $c_{ij} = a_i$, for $i > j$, can be solved in $O(n^2)$ time.*

Proof. Let π be an arbitrary tour with the set of peaks P and the set of valleys V. Notice, that $|P| = |V|$. We claim that the length of the tour can be calculated as $c(\pi) = \sum_{i=1}^{i=n} a_i - \sum_{j \in P} a_j + \sum_{k \in V} a_k$. Indeed, since the matrix is symmetric, all arcs can be considered as oriented from the smaller index to the bigger index, or, from a valley to a peak. It means that for each valley k, the corresponding length a_k should be included in the total length of the tour with the coefficient 2, and none of the values a_j, where j is a peak, is included. This proves the claim.

The considered special case of the TSP is thus equivalent to the problem of finding a tour $\pi^\star$ with set of peaks $P^\star$ and set of valleys $V^\star$ ($|P^\star|=|V^\star|$) such that $\sum_{k \in V^\star} a_k - \sum_{j \in P^\star} a_j$ is minimal.

Given a tour with a set of peaks P and a set of valleys V, there always exists a tour with the same sets of peaks and valleys such that the peaks and valleys are placed in increasing order in the tour. This means that, without loss of generality, it is possible to restrict the search for an optimal tour to the tours where the peaks and valleys appear in ordered sequence. Thus we may represent the sets P and V as two monotone sequences of indices. Since n is always a peak and 1 is always a valley, we exclude these two indices from further consideration. Let $f[p,v] = \min_{P,V}\{\sum_{k \in V} a_k - \sum_{j \in P} a_j | P \subseteq \{p, \ldots, n-1\}, V \subseteq \{v, \ldots, n-2\}, |P| = |V|\}$. We claim that the optimal sets $P^\star \setminus \{n\}$ and $V^\star \setminus \{1\}$ can be found in $O(n^2)$ time as the sets corresponding to $f[3,2]$. The latter value can be calculated using the following recursions

$$
\begin{aligned}
f[p,v] &= \min \begin{cases} f[p+1,v], \\ f[p,v+1], \\ a_v - a_p + f[p+1,v+1] \end{cases} \\
&\qquad p = 3,4,\ldots,n-2;\ v = p-1, p-2, \ldots, 2; \\
f[n-1,v] &= \min \begin{cases} f[n-1,v+1], \\ a_v - a_{n-1} \end{cases} \\
&\qquad v = 2,3,\ldots,n-2; \\
f[p,p] &= \infty, \\
&\qquad p = 3,4,\ldots,n-1.
\end{aligned}
$$

The recursions above do not prevent an index i from appearing in both sets P and V. This obstacle, however, can easily be overcome. Notice that including an index i into both sets does not change the value of the objective function. On the other hand, removing all such indices from both sets will result in sets that can be implemented as set of peaks and set of valleys, respectively.

4 TSP with Relaxed Kalmanson Matrices.

4.1 A symmetric $n \times n$ matrix $C = (c_{ij})$ is called a *Relaxed Kalmanson* matrix (RK-matrix) if it satisfies the four-point condition $\mathcal{B} \geq \mathcal{C}$, i.e. for all indices i,j,k,l, with $1 \leq i < j < k < l \leq n$, the inequalities $c_{ik} + c_{jl} \geq c_{il} + c_{jk}$ hold. The TSP with an RK-matrix will be called the RK-TSP. Cyclic permutation π will be called an $\mathcal{N}$*-permutation*, if it does not contain pairs of arcs $(i, \pi(i))$, $(j, \pi(j))$ such that either $i < \pi(i)$, $j > \pi(i) > \pi(j) > i$, or $i < \pi(i)$, $\pi(i) > j > i > \pi(j)$.

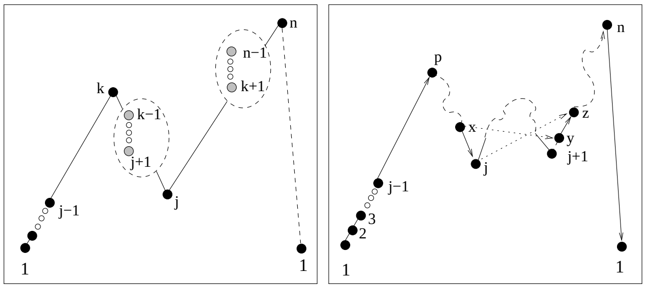
Proposition 4.1. *An optimal tour for the* RK-*TSP can be found among the $\mathcal{N}$-permutations.*

Proof. Let $\pi = \langle 1, \ldots, i, \pi(i), \ldots, j, \pi(j), \ldots \rangle$ with $i < \pi(i)$ and $j > \pi(i) > \pi(j) > i$. We transform π into $\pi_1 = \langle 1, \ldots, i, j, \pi^{-1}(j), \ldots, \pi(i), \pi(j), \ldots \rangle$ by removing arcs $(i, \pi(i))$ and $(j, \pi(j))$, reversing subsequence $\langle \pi(i), \ldots, j \rangle$ into $\langle j, \pi^{-1}(j), \ldots, \pi(i) \rangle$, and adding two new arcs (i,j) and $(\pi(i), \pi(j))$. It follows from $\mathcal{B} \geq \mathcal{C}$ that $c(\pi) \geq c(\pi_1)$ in this case.

Proposition 4.2. *Every $\mathcal{N}$-permutation contains the edge $(1,n)$.*

Proof. Let $\pi = \langle 1, \ldots, i, n, j, \ldots \rangle$, with $1 < j < i$. There always exists an arc (x,y) in the path from 1 to i with $x < j < y$, so that $\pi = \langle 1, \ldots, x, y, \ldots, i, n, j, \ldots \rangle$. We transform π into $\pi_1 = \langle 1, \ldots, x, n, \pi^{-1}(n), \ldots, y, j, \pi(j), \ldots \rangle$ by removing arcs (x,y) and (n,j), reversing subsequence $\langle y, \pi(y), \ldots, n \rangle$ into $\langle n, \pi^{-1}(n), \ldots, y \rangle$, and adding two new arcs (x,n) and (y,j). It follows from $\mathcal{B} \geq \mathcal{C}$ that $c(\pi) \geq c(\pi_1)$.

Proposition 4.3. *A structure of the path from 1 to n in an $\mathcal{N}$-permutation, to which we will refer as the $\mathcal{N}$-structure, can recursively be defined as follows: If there is no valley on the path from 1 to n, then this is the path* $\langle \mathbf{1}, 2, \ldots, n-1, \mathbf{n} \rangle$. *Otherwise let j be the minimal valley in the path from 1 to n. In this case the path has the structure $\langle \mathbf{1}, 2, \ldots, j-1, k, \{j+1, j+2, \ldots, k-1\}, j, \{k+1, k+2, \ldots, n-2, n-1\}, \mathbf{n} \rangle$ where k is a peak and the two paths – from j to k through the set $\{j+1, j+2, \ldots, k-2, k-1\}$, and from j to n through the set $\{k+1, k+2, \ldots, n-2, n-1\}$, have the $\mathcal{N}$-structure.*



(a) (b)

Figure 2: Illustration to $\mathcal{N}$-permutations: (a) definition of $\mathcal{N}$-structure; (b) transformation step.

In the proposition above, the sets are meant to be empty if the first index in the set is bigger than the last one. Figure 2(a) illustrates the definition of the $\mathcal{N}$-structure. For example, in the permutation $\langle 1, 14, 10, 12, 11, 13, 3, 6, 5, 4, 8, 7, 9, 2, 15, 1\rangle$, path $\langle 1, \dots, 15\rangle$ has 2 as the minimal valley, so $j = 2$ and $k = 14$ in this case; for the path $\langle 2, \dots, 14\rangle$ the corresponding pair (j, k) is $(3, 9)$, and so on.

Proof. Since every $\mathcal{N}$-permutation contains the edge $(1, n)$, and j is the minimal valley in the permutation, the permutation contains the subpath $\langle 1, 2, ..., j-2, j-1\rangle$. First we transform the part of the permutation which constitutes the path from $j-1$ to j. Let p be the maximal peak on the path from $j-1$ to j (there exists a peak, since j is a valley). By the same reasoning as in Proposition 4.2, the path is transformed into the path which contains the edge $(j-1, p)$.

If the path from p to j in the new permutation contains all indices of the set $P = \{j+1, j+2, \dots, p-2, p-1\}$, then the proposition is proved. Otherwise, consider the smallest index in P which is not on the path from p to j. Let it be $j+1$, which is a valley in this case (see Figure 2(b)). Consider an edge (x, j), $x = \pi^{-1}(j)$. There must be an edge (y, z) on the path from $j+1$ to n such that $j+1 \leq y < x < z$. We transform the permutation by replacing the edges (x, j) and (y, z) by the edges (x, y) and (j, z). Now the path from $j-1$ to valley j contains index $j+1$. There could be the case that the maximal peak on the path from $j-1$ to valley j has been changed. In this case we have to repeat transformations from Proposition 4.2. On each step we add a new index to the interval $j, j+1, \dots$, this guarantees the convergence of the process. After a finite number of steps the path from peak p to valley j consists only of indices $\{j+1, j+2, \dots, p-2, p-1\}$. The proposition is proved.

It follows from the definition of the $\mathcal{N}$-permutations that they belong to the set of twisted permutations [2]. Therefore an optimal tour could be found in $O(n^7)$ time by an algorithm from [7]. This complexity can be improved significantly as shown in the proposition below.

PROPOSITION 4.4. *The* RK-*TSP can be solved in* $O(n^4)$ *time.*

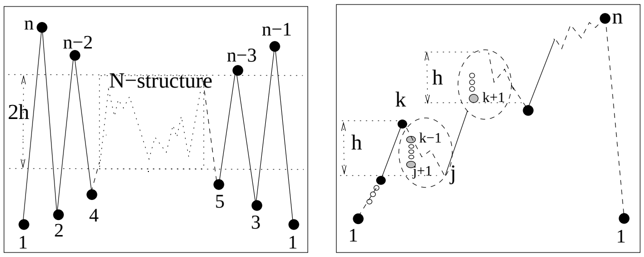
Proof. Let $L[p, q]$ be the length of the shortest path with the $\mathcal{N}$-structure from index p to index q through the set of indices $\{p+1, p+2, ..., q-2, q-1\}$, $p < q$, and $V[j, p, q]$ be the length of the shortest path with $\mathcal{N}$-structure from index j to index q through the set of indices $\{p, p+1, ..., q-2, q-1\}$, $j < p < q$. It follows from the definition of the $\mathcal{N}$-structure that the values L and V satisfy the following recursions:

$$L[p,q] = \min \begin{cases} \displaystyle\sum_{t=p}^{q-1} c_{t,t+1} \\ \displaystyle\min_{j<k} \Big\{ \sum_{t=p}^{j-2} c_{t,t+1} + c_{j-1,k} + L[j,k] + \\ \qquad\qquad + V[j, k+1, q] \Big\} \end{cases}$$

$$V[j,p,q] = \begin{cases} c_{jp}, & \text{if } p > q, \\ min \begin{cases} c_{jp} + L[p,q] \\ \displaystyle\min_k \Big\{ c_{jk} + L[p,k] + \\ \qquad\qquad + V[p, k+1, q] \Big\} \end{cases} \end{cases}$$

If in the formulae above the upper limit in a sum is smaller than the lower limit, then the sum is zero. The length of the optimal tour can be calculated as $L[1, n] + c_{n1}$. Each of the values $L[\ ]$ can be calculated in $O(n^2)$ time, each of the values $V[\ ]$ can be calculated in linear time. An overall time complexity of $O(n^4)$ results.

4.2 One of the reasons why exponential neighborhoods are interesting is the fact that they can be used in local search algorithms. The complexity $O(n^4)$ seems pretty high to make the RK-neighborhood useful for these purposes. To address this concern, below we characterize related neighborhoods which are searchable in linear time. We also provide classes of distance matrices for which an optimal tour can be found in the respective neighborhoods.



(a) (b)

Figure 3: Specially structured tours: (a) structure of tours from the neighborhood searchable in $O(n+h^4)$ time; (b) structure of tours from the neighborhood searchable in $O(nh^3)$ time.

Corollary 4.1. *If an* RK-*matrix* (c_{ij}) *satisfies conditions* $c_{ji}+c_{kl} \leq c_{jl}+c_{ki}$ *for all* $i<j,\ j+h \leq k<l$, *where* h *is a given constant, then the* RK-*TSP can be solved in* $O(nh^3)$ *time.*

Proof. Using the tour improvement technique, it is easy to show that the additional property of the distance matrix prevents long arcs $(j-1,k)$ from appearing in an optimal tour (see Figures 2(a) and 3(b)): k here cannot be bigger than $j+h$. Incorporating this observation into the dynamic programming recursions leads to a time complexity of $O(nh^3)$ and to an optimal solution being a specially structured tour from the set of $\mathcal{N}$-permutations.

Proposition 4.5. *If an* RK-*matrix* (c_{ij}) *satisfies conditions* $c_{ji}+c_{kl} \geq c_{jl}+c_{ki}$ *for all* $i<j,\ j+h \leq k<l$, *where* h *is a given constant, then an optimal tour for the* RK-*TSP can be found among the tours that have all indices* $s,\ s>\lfloor (n-1)/2 \rfloor + h$, *as peaks, and all indices* $t,\ t<\lceil (n-1)/2 \rceil - h+2$, *as valleys.*

Proof. A pair of edges (i,j) and (k,l) in a tour τ is called non-crossing, if $i<j,\ j+h<k<l$. Notice, that we use the term "edge" here, meaning that the orientation does not matter. First, we show that an optimal tour can be found among the tours without non-crossing edges.

Suppose that edges (i,j) and (k,l) have the same orientation, that is $j=\tau(i)$ and $l=\tau(k)$ (case of $i=\tau(j)$ and $k=\tau(l)$ is symmetric). It follows then from the inequalities $c_{ji}+c_{kl} \geq c_{jl}+c_{ki}$ that deleting edges (i,j) and (k,l) and introducing two new edges (i,k) and (j,l) will not increase the length of the tour, and eliminates at the same time two non-crossing edges.

In the case of $j=\tau(i)$ and $k=\tau(l)$, we delete these two edges and introduce two new edges (i,l) and (j,k). The change in the length of the tour can be calculated as $c_{il}+c_{kj}-c_{ij}-c_{kl} = c_{il}+c_{jk}-c_{ij}-c_{kl}+c_{ik}-c_{ik}+c_{jl}-c_{jl} = (c_{ik}+c_{jl}-c_{ij}-c_{kl})-(c_{ik}+c_{jl}-c_{il}-c_{jk})$. It follows then from the inequalities $c_{ji}+c_{kl} \geq c_{jl}+c_{ki}$ and from the fact that (c_{ij}) is an RK-matrix, that the transformation above will not increase the length of the tour.

Suppose now that we have a tour τ without non-crossing edges and there is an edge (x,j), $x<j$, in the tour. It means that any index y, $y \geq j+h$, can only be a peak in the tour (otherwise, (x,j) and (y,z), $z>y$, would be a pair of non-crossing edges). So, we have at least $n-j-h+1$ peaks in the tour. If j is also a peak, then, taking into account the fact that the total number of peaks cannot be bigger than $\lfloor n/2 \rfloor$, we have the following inequality for a maximal possible value of j: $j \geq n-\lfloor n/2 \rfloor - h+2 = \lceil n/2 \rceil - h+2$. So, all indices t, $t<\lceil n/2 \rceil - h+2$ are valleys in the tour (there are no edges (x,t) with $x<t$). If index j is not a peak, then the total number of peaks in the tour cannot be bigger than $\lfloor (n-1)/2 \rfloor$. This yields the following inequality for maximal possible value of j: $j \geq \lceil (n-1)/2 \rceil - h+2$.

Summarizing, all indices t, $t<min\{\lceil n/2 \rceil - h+2, \lceil (n-1)/2 \rceil - h+2\} = \lceil (n-1)/2 \rceil - h+2$ are valleys in the tour τ.

Using the same reasoning, it can be shown that all indices s, $s>\lfloor (n-1)/2 \rfloor + h$, are peaks in the tour. This completes the proof.

Corollary 4.2. *If an* RK-*matrix* (c_{ij}) *satisfies conditions* $c_{ji}+c_{kl} \geq c_{jl}+c_{ki}$ *for all* $i<j,\ j+h \leq k<l$, *where* h *is a given constant, then the* RK-*TSP can be solved in* $O(n+h^4)$ *time.*

Proof. Positions of all indices s, $s>\lfloor (n-1)/2 \rfloor + h$, which are peaks, and all indices t, $t<\lceil (n-1)/2 \rceil - h+2$, which are valleys, are identified according to the $\mathcal{N}$-structure: valley 2 precedes peak n, peak $n-1$ follows valley 1 and precedes valley 3, peak $n-2$ precedes valley 2, and so on (see Figure 3(a)). There are only $O(h)$ indices left the positions of which are not fixed. An optimal placement of these indices can be found using the recursions for finding an optimal $\mathcal{N}$-structure, confer Proposition 4.4.

Corollary 4.3. *For the TSP with an* $n \times n$ *distance matrix* (c_{ij}) *satisfying conditions* $\mathcal{A} \geq \mathcal{B}$, $\mathcal{B} \geq \mathcal{C}$ *there exists an optimal tour within the following small set of tours: these are specially structured tours* $\sigma_{Smax1}, \sigma_{Smax2}$ *for* n *even, and tours* $\sigma_{Smax}, \sigma_{Smax3}, \sigma_{Smax4}$, *for* n *odd.*

Proof. A distance matrix (c_{ij}) satisfying conditions $\mathcal{A} \geq \mathcal{B}$, $\mathcal{B} \geq \mathcal{C}$ is an RK-matrix satisfying additional constraints $c_{ij}+c_{kl} \geq c_{il}+c_{kj}$ for $i<j<k<l$. The latter inequalities yield inequalities $c_{iu}+c_{kl} \geq c_{il}+c_{ku}$ for $u<i<i+2 \leq k<l$. Indeed, this fact

follows immediately from the simple transformations $c_{iu} + c_{kl} - c_{il} - c_{ku} = c_{iu} + c_{kl} - c_{il} - c_{ku} + c_{ij} - c_{ij} + c_{kj} - c_{kj} = (c_{ij} + c_{kl} - c_{il} - c_{kj}) + (c_{iu} + c_{kj} - c_{ij} - c_{ki}) = (c_{ij} + c_{kl} - c_{il} - c_{kj}) + (c_{ui} + c_{jk} - c_{ji} - c_{ik})$, where j is an index between i and k: $i < j < k$ (existence of j follows from the inequality $i + 1 < k$).

It follows from Proposition 4.5, that only for small number of indices, their positions in an optimal tours are not finalized. For n even, $n = 2m$, these are indices m and $m + 1$. There are only two possibilities: either m is a valley and $m + 1$ is a peak, or both are intermediate indices. These two options lead to two tours, $\sigma_{\mathrm{S}max1}$ and $\sigma_{\mathrm{S}max2}$: $\sigma_{\mathrm{S}max1} = \langle 1, n, 2, \ldots, m + 3, m-1, m+1, m, m+2, m-2, \ldots, 3, n-1, 1\rangle$, $\sigma_{\mathrm{S}max2} = \langle 1, n, 2, \ldots, m+3, m-1, m, m+1, m+2, m-2, \ldots, 3, n-1, 1\rangle$ if m is odd, and $\sigma_{\mathrm{S}max1} = \langle 1, n, 2, \ldots, m - 2, m + 2, m, m + 1, m - 1, m - 2, \ldots, 3, n - 1, 1\rangle$, $\sigma_{\mathrm{S}max2} = \langle 1, n, 2, \ldots, m-2, m+2, m+1, m, m+2, m-2, \ldots, 3, n-1, 1\rangle$ if m is even.

For n odd, $n = 2m + 1$, positions of indices $m, m + 1, m + 2$ are not yet finalized. If m is a valley and $m + 2$ is a peak, then it gives the tour σ_{Smax}, which is also an optimal tour for the Supnick MaxTSP. There are other two options to place indices $m, m + 1, m + 2$ as prescribed by the $\mathcal{N}$-structure. These two options lead to two tours $\sigma_{\mathrm{S}max3}$ and $\sigma_{\mathrm{S}max4}$: $\sigma_{\mathrm{S}max3} = \langle 1, \ldots, m + 3, m + 2, m, m + 1, m - 1, \ldots, 1\rangle$, $\sigma_{\mathrm{S}max4} = \langle 1, \ldots, m + 3, m + 1, m + 2, m, m - 1, \ldots, 1\rangle$.

5 TSP with Relaxed Supnick Matrices

A symmetric $n \times n$ matrix $C = (c_{ij})$ is called a *Relaxed Supnick* matrix (RS-matrix) if it satisfies the four-point condition $\mathcal{B} \le \mathcal{C}$, i.e. for all indices i, j, k, l, $1 \le i < j < k < l \le n$, the inequalities $c_{ik} + c_{jl} \le c_{il} + c_{jk}$ hold. The TSP with an RS-matrix will be called the RS-TSP. Two well known examples related to the RS-TSP are the Supnick TSP ($\mathcal{B} \le \mathcal{C}$, $\mathcal{A} \le \mathcal{B}$) and the Kalmanson MaxTSP ($\mathcal{B} \le \mathcal{C}$, $\mathcal{A} \ge \mathcal{B}$). The optimal tour σ_{Smin} for the Supnick TSP has the property that indices in the tour are evenly spread on the slopes of the tour. Motivated by this observation, we introduce the following definition. The cyclic permutation π is called a *balanced* permutation, if it does not contain pairs of arcs $(i, \pi(i))$, $(j, \pi(j))$ such that either $i < \pi(j) < j < \pi(i)$, or $i > \pi(j) > j > \pi(i)$; and it does not contain chains $i, \pi(i), \pi^2(i)$ and $j, \pi(j), \pi^2(j)$ such that either $i < j < \pi(j) < \pi(i) < \pi^2(i) < \pi^2(j)$ or $i > j > \pi(j) > \pi(i) > \pi^2(i) > \pi^2(j)$.

Proposition 5.1. *An optimal tour for the* RS-*TSP can be found among the balanced permutations.*

Proof. The proof is based on the tour-improvement technique. Suppose that there is a pair of arcs $(i, \pi(i))$, $(j, \pi(j))$ such that $i < \pi(j) < j < \pi(i)$. It is easy to see that deleting these two arcs and introducing two new arcs (i, j) and $(\pi(i), \pi(j))$ will not increase the length of the tour. For the chains $i, \pi(i), \pi^2(i)$ and $j, \pi(j), \pi^2(j)$ with either $i < j < \pi(j) < \pi(i) < \pi^2(i) < \pi^2(j)$, or $i > j > \pi(j) > \pi(i) > \pi^2(i) > \pi^2(j)$, the corresponding transformation of the tour is a swap of $\pi(i)$ and $\pi(j)$.

The status of the RS-TSP remains unknown: we do not know whether it is possible to find an optimal balanced tour in polynomial time. We can, however, describe an alternative characterization of balanced tours. This characterization can be used to introduce new exponential neighborhoods and to identify corresponding structures in the distance matrix for new solvable cases. To simplify the situation, we consider a special subset of the balanced tours and describe additional constraints on the distance matrix when an optimal tour can be found in this special subset.

We will consider partially constructed tours on the sets of indices $\{1, 2, 3, \ldots, m - 1, m\}$ with $m = 1, 2, \ldots, n$. For a fixed m, a partially constructed tour will consist of a set of subsets of indices; indices from each subset are placed in the tour on consecutive positions. We refer to each of these subsets $\{i_1, \ldots, j_1\}$ as *fragment* $[i_1, j_1]$, stressing that i_1 is the first, and j_1 is the last element in the corresponding sub-sequence. Notice that it is not necessarily the case that fragment $[i_1, j_1]$ with $i_1 < j_1$ contains, for example, $i_1 + 1$. For a one element fragment we still use the same notation $[i, i]$. For example, if we start with a tour, where 1 and 2 are two valleys, this initial tour can be represented by the two fragments $[1, 1]$ and $[2, 2]$.

Mutual placement of fragments is not fixed, i.e., they can be permuted. The fragments can also be inverted, i.e. fragment $[i, j]$ can be replaced by the fragment $[j, i]$.

Definition 5.1. *A tour will be called a strongly balanced tour if it can be constructed using the procedure described below.*

Start with an initial tour $[1, 1]$ *and repeat the next step for* $m = 2, \ldots, n - 1$*:*

- *Given a partial tour on the set of indices* $\{1, \ldots, m - 1\}$*, the tour is represented by the fragments* $[i_1, j_1]$, $[i_2, j_2]$, $\ldots$, $[i_p, j_p]$. *Let* $i_{min1} = min\{i_1, j_1, i_2, j_2, \ldots, i_p, j_p\}$, $i_{min2} = min\ \{i_1, j_1, i_2, j_2, \ldots, i_p, j_p\} \setminus \{i_{min1}\}$, $i_{min3} = min\ \{i_1, j_1, i_2, j_2, \ldots, i_p, j_p\} \setminus \{i_{min1}, i_{min2}\}$. *Choose one of the options below and add index* m *to the partial tour:*
  - *m is placed as a new valley; this creates a new fragment* $[m, m]$*;*

- *m is placed as an intermediate index and is adjacent to i_{min1}; fragment $[i_{min1}, s]$ in the partially constructed tour is replaced by the new fragment $[m, s]$ in this case.*
- *m is placed as a new peak merging two fragments; m is adjacent to i_{min1} and to either i_{min2} or to i_{min3}. In the first case, the fragments $[i_{min1}, j]$ and $[i_{min2}, s]$ are merged into $[j, s]$, in the second case, the fragments $[i_{min1}, i_{min2}]$ and $[i_{min3}, s]$ are merged into $[i_{min2}, s]$.*

The last node n can be added only to a partial tour consisting of one fragment.

It can be shown that the set of strongly balanced tours is a special subset of the balanced tours. In particular, it can be shown that the definition above excludes from consideration tours that contain either an arc $(i, \pi(i))$ and chain $j, \pi(j), \pi^2(j)$ with $i < j < \pi(j) < \pi(i) < \pi^2(j)$, or an arc $(i, \pi(i))$ and chain $j, \pi(j), \pi^2(j)$ with $j > i > \pi(j) > \pi^2(j) > \pi(i)$.

Definition 5.2. *An RS-matrix (c_{ij}) which satisfies additional constraints $c_{il} + c_{jk} + c_{km} \geq c_{ik} + c_{jm} + c_{kl}$ for all $i < j < k < l < m$ is called strong RS-matrix.*

It can be shown that the system of inequalities above is equivalent, for an RS-matrix (c_{ij}), to the system of inequalities $c_{j-1,k} + c_{j,k-1} + c_{k-1,n} \geq c_{j-1,k-1} + c_{jn} + c_{k-1,n}$, and therefore can be checked in $O(n^2)$ time.

Proposition 5.2. *An optimal tour for the TSP restricted to the class of strong RS-matrices can be found within the set of strongly balanced tours.*

The status of the TSP with a strong RS-matrix still remains unknown. We need a bit more structure in an RS-matrix in order to identify a new polynomially solvable case, as shown in the proposition below.

Corollary 5.1. *If a strong RS-matrix (c_{ij}) satisfies conditions $c_{ji} + c_{kl} \leq c_{jl} + c_{ki}$ for all $i < j$, $j + h \leq k < l$, where h is a given constant, then the TSP with such matrix can be solved in polynomial time.*

Sketch of the proof. It can be shown that the additional property of the distance matrix prevents pairs of arcs (i, k) and (j, l) with $j + h \leq k$ $(i < j, k < l)$ from appearing in an optimal tour. For the balanced tours this is possible only, if the number of fragments p on each step of the construction procedure in Proposition 5.2 does not exceed a constant $d = \lfloor h/2 \rfloor + 1$. It is shown in [20] that an optimal tour with this structure can be found in $O(n^{d+1})$ time. This proves the corollary.

The algorithm in [20] does not use the additional property of an optimal tour being strongly balanced. Careful analysis of balanced tours with a bounded number of fragments may lead to algorithms with much better performance as illustrated in the statement below.

Corollary 5.2. *If a strong RS-matrix (c_{ij}) satisfies conditions $c_{ji} + c_{kl} \leq c_{jl} + c_{ki}$ for all $i < j$, $j + 3 \leq k < l$, then an optimal TSP tour can be found in $O(n)$ time among $\Theta(2^n)$ strongly balanced tours.*

Sketch of the proof. An optimal tour for the corresponding TSP can be found among the strongly balanced tours containing no more than two fragments at each step of the construction procedure described in Definition 5.1. Construction of any of these tours starts either from a partial tour (fragment) $[1, 2]$, or from a tour with two valleys – 1 and 2 (fragments $[1, 1]$ and $[2, 2]$). There are two options to extend each of these fragments by adding index 3.

For fragment $[1, 2]$, index 3 can either be adjacent to 1 (it gives a new fragment $[2, 3]$, to which we will refer to as fragment of type F1), or be defined as a new valley (it gives the fragment $[1, 2]$, $[3, 3]$ of a new type, to which we will refer as type F2).

For the initial tour $[1, 1]$, $[2, 2]$, index 3 can either be adjacent to 1 (it gives the fragments $[1, 3]$,$[2, 2]$ of type F3), or be defined as a new peak (it gives the fragment $[1, 3, 2]$ of type F4). In total, there are thirteen possible types of patterns listed in Figure 4. These fragments can be combined together in a way which can easily be used in dynamic programming recursions for finding an optimal strongly balanced tour. Schematic representation of the relations among all possible types of the fragments is represented as a rooted tree in Figure 4.

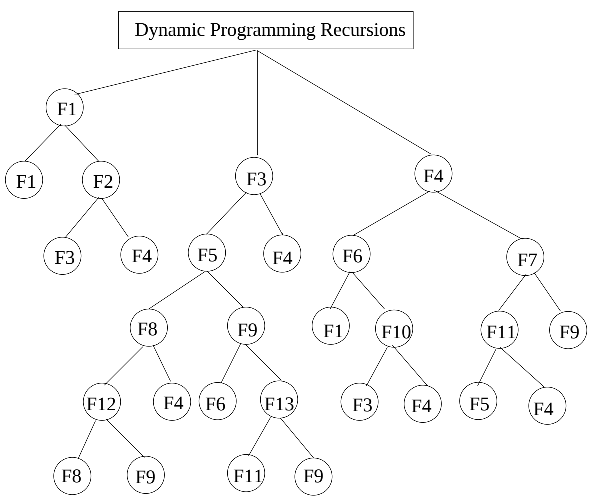
Given a constant h, a procedure described in Proposition 5.2 can be used in a computer program to identify types of possible tour fragments and the relationship among them to be used in a dynamic programming algorithm. To our best knowledge, this idea to first identify relationships to be used in a dynamic programming recursions by a computer-based approach and to use the obtained relationships in a second step in a dynamic programming routine as input, is new in the area of combinatorial optimization.

6 Conclusions

In this note we have classified all possible four-point conditions for the symmetric TSP. Our analysis allowed us

Tour Fragments	
F1	$[i, i+1]$
F2	$[i, i+1]$, $[i+2, i+2]$
F3	$[i, i+2]$, $[i+1, i+1]$
F4	$[i, \ldots, i+2, \ldots, i+1]$
F5	$[i, i]$, $[i+1, i+2]$
F6	$[i, i+2]$
F7	$[i, i+1]$, $[i+3, i+3]$
F8	$[i, i+3]$, $[i+1, i+2]$
F9	$[i, \ldots, i+3, \ldots, i+2]$
F10	$[i, i+2]$, $[i+3, i+3]$
F11	$[i, i+3]$, $[i+2, i+2]$
F12	$[i, i+1]$, $[i+2, i+3]$
F13	$[i, i+2]$, $[i+4, i+4]$

(a)



(b)

Figure 4: Schematic representation of dynamic programming recursions: (a) list of possible types of fragments; (b) relationship among the types of fragments in the dynamic programming recursions.

to describe families of new exponential neighborhoods searchable in polynomial time. Given, that a distance matrix satisfies specified conditions, an optimal TSP tour can be found in one of these neighborhoods.

There is a nice connection of the four-point conditions with the well known 2-*opt* heuristic (see, e.g. [12]). Given a TSP tour, the 2-*opt* heuristic looks for a pair of arcs (j, l) and (i, k) such that $c_{ji} + c_{kl} < c_{jl} + c_{kj}$. If such a pair is found, then the tour is transformed into the new tour by replacing arcs the (j, l) and (i, k) with the pair of new arcs (i, j) and (k, l). If we consider distance matrices such that the 2-*opt* procedure always improves the tour, if $1 \leq i < j < k < l \leq n$, then this is a specially structured matrix which satisfies the four-point condition $\mathcal{A} < \mathcal{B}$. If we fix i to be the minimal element in the 4-tuple $\{i, j, k, l\}$ and consider all 6 possible ordering for the triple of indices $\{j, k, l\}$ and corresponding conditions on the distance matrix, then we will end up with the main four-point conditions discussed in this paper.

The straightforward generalization is to consider the 3-*opt* procedure and corresponding six point conditions. The obvious difficulty is the number of all possible six point conditions, which is 480. If we add all possible pairs of the conditions, then we end up with 115440 cases to be considered and analyzed. A potential approach to cope with this vast number of cases would be to use computer support to classify the cases. The computer based technique discussed in the previous section (which is to be used for generating dynamic programming recursions) is a first example of a possible computer based approach to research in this exciting area.

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