

Chapter 2

(GENERALIZED) CONVEXITY AND DISCRETE OPTIMIZATION

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Abstract This short survey exhibits some of the important roles (generalized) convexity plays in integer programming. In particular integral polyhedra are discussed, the idea of polyhedral combinatorics is outlined and the use of convexity concepts in algorithmic design is shown. Moreover, combinatorial optimization problems arising from convex configurations in the plane are discussed.

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1. Introduction

Convexity plays a crucial role in many areas of mathematics. Problems which show convex features are often easier to solve than similar problems in general. This short survey based on personal preferences intends to exhibit some of the roles convexity plays in discrete optimization. In the next section we discuss convex polyhedra all of whose vertices have integral coordinates. In Section 3 we outline the concept of polyhedral combinatorics which became basic for solving \mathcal{NP} -hard

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problems like the travelling salesman problem. In Section 4 we show some of the roles (generalized) convexity plays in the algorithmic design for combinatorial optimization problems. In the last section combinatorial optimization problems arising from convex geometric configurations will be discussed.

2. Convexity and integer programming

At the end of the 19th century Minkowski began to study convex bodies which contain lattice points. In 1893 he proved the following fundamental theorem (see also his monograph *Geometry of Numbers* of 1896):

Theorem 2.1 *Let C be a convex body in \mathbb{R}^d , symmetric with respect to the origin, and let the volume $V(C)$ of C be $V(C) \geq 2^d$. Then C contains a pair of points with integral coordinates.*

In connection with the development of linear and integer programming this area of the *geometry of numbers* got a new relevance. The main theorem of linear programming states that the finite optimum of a linear program is always attained in an extreme point (vertex) of the set of feasible solutions. If we can derive a bound on the coordinates of vertices of the feasible set, even if the underlying polyhedral set is unbounded, then the feasibility and optimality of an integer program can be checked in finitely many steps. To be more precise, let us assume that A is an integral $m \times n$ matrix and let $b \in \mathbb{Z}^m$. We consider the points with integral coordinates in the convex polyhedral set

$$P := \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$$

and call $S := P \cap \mathbb{Z}^n$. The following theorem, see Nemhauser and Wolsey (1988) Theorem I.5.4.1., is basic that an integer programming problem can be solved by enumeration.

Theorem 2.2 *Let $K := \max_{1 \leq i \leq m, 1 \leq j \leq n} (|a_{ij}|, |b_i|)$. If x is an extreme point of $\text{conv}(S)$, then*

$$0 \leq x_j \leq ((m+n)nK)^n \text{ for } j = 1, 2, \dots, n.$$

As a consequence of this result the feasibility and optimality problems in integer linear programming belong to the complexity class \mathcal{NP} . Bank and Mandel (1988) generalized this result to constraint sets described by quasi-convex polynomials with integer coefficients.

Since integer programming can be reduced to linear programming provided that all extreme points of the feasible region have integral coordinates, there is a special interest in convex polyhedral sets with integral

vertices. A convex polyhedron

$$P := \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$$

is called *integral*, if all its vertices have integral coordinates. A nice characterization of integral polyhedral sets defined by arbitrary right hand sides b has been given by Hoffman and Kruskal (1956). A matrix A is called *totally unimodular*, if any regular submatrix of A has determinant ± 1 . Now the following fundamental theorem holds:

Theorem 2.3 (Hoffman and Kruskal, 1956)

Let A be an integral matrix. Then the following two statements are equivalent:

1 $P(A, b) := \{x \in \mathbb{R}^n \mid Ax \leq b, x \geq 0\}$ is integral for all b with $P(A, b) \neq \emptyset$.

2 A is totally unimodular.

Important examples for problems with totally unimodular coefficient matrices are assignment problems, transportation problems and network flow problems. Seymour (1980) showed that totally unimodular matrices can be recognized in polynomial time.

If we specialize the right hand side in the constraint set to b with $b_i = 1$ for all i , we get the constraint sets of

- set packing problems: $Ax \leq 1, x \geq 0$,
- set partitioning problems: $Ax = 1, x \geq 0$,
- set covering problems: $Ax \geq 1, x \geq 0$.

For this kind of problems not only totally unimodular matrices, but even a larger class of matrices leads to integral polyhedra. We call a matrix A with entries 0 and 1 *balanced*, if it does not contain a square submatrix of odd order with row and column sums equal to 2. For example, the following 3×3 submatrix constitutes a forbidden submatrix:

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}$$

Fulkerson, Hoffman and Oppenheim (see Fulkerson et al (1974)) showed the following result.

Theorem 2.4 If A is balanced, then the set partitioning problem

$$\min\{c'x \mid Ax = 1, x \geq 0\}$$

has integral optimal solutions.

For many years the recognition of balanced matrices has been an open problem. In 1999, Conforti, Cornuéjols and Rao (see Conforti et al. (1999)) showed that balanced matrices can be recognized in polynomial time.

The following result of Berge (1972) with respect to set packing and set covering problems is more along the lines of the Hoffman-Kruskal theorem.

Theorem 2.5 *Let matrix A be without 0-row and 0-column. Then the following statements are equivalent:*

- 1 A is balanced.
- 2 $\{x | Ax \leq b, x \geq 0\}$ is integral for all b with $b_i = 1$ or ∞ .
- 3 $\{x | Ax \geq b, x \geq 0\}$ is integral for all b with $b_i = 1$ or 0 .

For a recent survey on packing and covering problems the interested reader is referred to Cornuéjols (2001).

3. Polyhedral combinatorics

In the following we consider combinatorial optimization problems which can be described by

- a finite ground set E ,
- a class \mathcal{F} of feasible solutions which are subsets $F \subseteq E$, and
- cost coefficients $c(e)$ for all elements $e \in E$.

The cost of a feasible solution F is defined by $c(F) := \sum_{e \in F} c(e)$. The goal is to find a feasible solution with minimum cost.

For example, the *travelling salesman problem* may be described by the ground set E consisting of all edges (roads) between n vertices (cities) of a graph. A feasible solution F corresponds to a tour through all cities. A *tour* is a subset of the edges which corresponds to a cyclic permutation ϕ of the underlying vertex set, i.e., F consists of all edges $[i, \phi(i)]$, $1 \leq i \leq |V|$. Less formally spoken, a tour visits all vertices of the graph starting from vertex 1 and does not visit any vertex twice. The length of a tour F is given by $c(F) := \sum_{e \in F} c(e)$. The objective is to find a tour with minimum length.

In order to model this problem with binary variables we introduce a 0-1 vector x with $|E|$ components. A feasible solution F corresponds to

$$x(F) := \begin{cases} 1, & \text{if } e \in F \\ 0, & \text{if } e \notin F. \end{cases}$$

The combinatorial optimization problem

$$\min_{F \in \mathcal{F}} \sum_{e \in F} c(e)$$

can be written as

$$\min c'x \text{ for } x \in \text{conv}\{x(F), F \in \mathcal{F}\}.$$

This means that the linear function $c'x$ is to be minimized over the convex hull of finitely many points. *Polyhedral combinatorics* consists in describing the polytopes given as convex hull of all feasible points by linear inequalities. Let us discuss as examples matching problems and symmetric travelling salesman problems.

Matching problems

A matching M is a subset of edges of an undirected, finite graph $G = (V, E)$ with vertex set V and edge set E where every vertex is incident with at most one edge of M . The maximum cardinality matching problem asks for a maximum matching in G , i.e., for a matching with a maximum number of edges. The ground set E contains the edges of G , feasible sets are the matchings M . We want to formulate the maximum cardinality matching problem as a binary linear program. To this end we introduce for each edge $j \in E$ a variable x_j . Let $\delta(v)$ denote the set of edges incident with vertex v . Then we get the following obvious necessary inequalities:

$$\begin{aligned} \sum_{j \in \delta(v)} x_j &\leq 1 \text{ for all } v, \\ x &\geq 0. \end{aligned}$$

If we consider the graph K_3 , i.e., the complete graph with three vertices and three edges (which form a triangle), then the vector $x = (1/2, 1/2, 1/2)$ fulfills the inequalities above, but does not correspond to a matching. Thus it is necessary to add additional constraints in the case of a non-bipartite graph. One can show that in the case of a bipartite graph the above mentioned constraints are sufficient for describing a matching. Let $\gamma(W)$ denote the subset of all edges with both endpoints in $W \subseteq V$. Edmonds (1965) introduced for the maximum cardinality matchings in non-bipartite graphs the additional constraints

$$\sum_{j \in \gamma(W)} x_j \leq 1/2(|W| - 1),$$

for all $W \subseteq V$ with $|W| \geq 3$, odd.

Theorem 3.1 (Edmonds, 1965)

The matching polytope is fully described by

$$\begin{aligned} x &\geq 0, \\ \sum_{j \in \delta(v)} x_j &\leq 1 \text{ for all } v \in V, \\ \sum_{j \in \gamma(W)} x_j &\leq 1/2(|W| - 1) \text{ for all } W \subseteq V \text{ with } |W| \geq 3, \text{ odd.} \end{aligned}$$

Symmetric travelling salesman problems

As a second example we consider the symmetric travelling salesman problem (TSP). Let again a finite, undirected graph $G = (V, E)$ with vertex set V and edge set E be given. In order to describe the feasible sets (tours) by linear inequalities we introduce a binary variable $x(e)$ for every edge $e \in E$. Obviously the following inequalities must be fulfilled:

$$0 \leq x(e) \leq 1, \text{ for all } e \in E, \quad (2.1)$$

and

$$\sum_{e \in \delta(v)} x(e) = 2, \text{ for all } v \in V. \quad (2.2)$$

But these inequalities do not fully describe tours, since they may be incidence vectors of more than one cycle in G , so-called *subtours*. Therefore one requires also the so-called *subtour elimination constraints*

$$\sum_{e \in \delta(W)} x(e) \geq 2 \text{ for all } W \subseteq V, 2 \leq |W| \leq |V| - 2. \quad (2.3)$$

Now one can show

Theorem 3.2 *The integral points lying in the convex polyhedron (2.1)-(2.3) correspond exactly to tours.*

It should be noted that a linear program with constraints (2.1)-(2.3) can be solved in polynomial time, even if there are exponentially many inequalities of the form (2.3). The convex polytope described by (2.1)-(2.3) may, however, have fractional vertices which do not correspond to tours. Thus further inequalities must be added which cut off such fractional vertices. There are many classes of such additional inequalities known, e.g. comb inequalities, clique tree inequalities and many others. The interested reader is referred to e.g. Grötschel, Lovász and Schrijver (see Grötschel et al. (1988)). It should be noted that a complete characterization of the convex hull of all tours is not known in general.

Since the polytope described by (2.1)-(2.3) may have non-integral extreme points, the following *separation problem* plays an important role for solving the TSP: If the optimal solution for the linear program with the feasible set (2.1)-(2.3) is not integral, we have to add a so-called *cutting plane*, i.e., a linear constraint which is fulfilled by all tours, but which cuts off the current infeasible point. Usually such a cutting plane is determined by heuristics and is taken from the class of comb inequalities, clique tree inequalities or other facet defining families of linear inequalities for the TSP polytope.

4. (Generalized) Convexity and algorithms

In this section we will point out that convexity also plays an important role in algorithms for solving a convex or linear integer program. Let $f_i(x)$, $1 \leq i \leq m$, be quasiconvex functions defined on a region $D \subseteq \mathbb{R}^n$ and consider the *convex integer program*

$$\text{minimize} \quad c'x \quad (2.4)$$

$$\text{subject to} \quad f_i(x) \leq 0 \quad 1 \leq i \leq m, \quad (2.5)$$

$$x \text{ integer.} \quad (2.6)$$

Branch and bound method

When we use a branch and bound method for solving (4)-(6), we first solve the underlying convex program without the constraint x being integral. If the solution x^* is integral, we are done. Otherwise, say, x_1^* is not integral. We create two new problems by adding either

$$x_1 \leq [x_1^*]$$

or

$$x_1 \geq [x_1^*] + 1.$$

Instead of solving these two subproblems we can - due to the *convexity of the level sets* - fix the variable x_1 to $[x_1^*]$ and $[x_1^*] + 1$, respectively. Therefore we solve a problem with $x_1 = [x_1^*]$ and a problem with $x_1 = [x_1^*] + 1$. Now assume that the solution of the first subproblem with the additional constraint $x_1 = [x_1^*]$ is still not integral. Then we must generate three new subproblems in the next branching step, namely two subproblems for fixing a new variable to an integer value and one subproblem with fixing x_1 to $[x_1^*] - 1$. For details, see e.g. Burkard (1972). Thus the convexity of the level sets helps to fix variables which accelerates the solution of the problem.

Cutting plane methods

Given problem (2.4)-(2.6), we first solve again the underlying convex

program without the constraint \mathbf{x} being integral. If the solution obtained in this way is not integral, we search for a valid inequality which cuts off this solution, but which does not cut off any feasible integral solution (separation problem). If no valid inequality can be found, we branch (*branch and cut method*). This method uses essentially the fact that the intersection of two convex sets is again convex.

Subgradient optimization

For hard combinatorial optimization problems often a strong lower bound can be computed by a Lagrangean relaxation approach which uses the minimization of a non-smooth convex function. Held and Karp (1971) used such an approach very successfully for the symmetric traveling salesman problem, see also Held et al. (1974). We will illustrate this approach by considering the axial 3-dimensional assignment problem.

The axial 3-dimensional assignment problem can be formulated in the following way:

$$\begin{aligned}
 \min \quad & \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n c_{ijk} x_{ijk} \\
 \text{s.t.} \quad & \sum_{j=1}^n \sum_{k=1}^n x_{ijk} = 1, \quad i = 1, 2, \dots, n, \\
 & \sum_{i=1}^n \sum_{k=1}^n x_{ijk} = 1, \quad j = 1, 2, \dots, n, \\
 & \sum_{i=1}^n \sum_{j=1}^n x_{ijk} = 1, \quad k = 1, 2, \dots, n, \\
 & x_{ijk} \in \{0, 1\} \text{ for all } 1 \leq i, j, k \leq n.
 \end{aligned}$$

Karp (1972) showed that this problem is \mathcal{NP} -hard. In order to compute strong lower bounds we take two blocks of the constraints into the objective function via Lagrangean multipliers:

$$L(\pi, \epsilon) :=$$

$$\min \left\{ \sum_{i=1}^n \sum_{j=1}^n \sum_{k=1}^n (c_{ijk} + \pi_j + \epsilon_i) x_{ijk} - \sum_{j=1}^n \pi_j - \sum_{i=1}^n \epsilon_i \right\}$$

such that

$$\begin{aligned}
 & \sum_{i=1}^n \sum_{j=1}^n x_{ijk} = 1, \quad k = 1, 2, \dots, n \\
 & x_{ijk} \in \{0, 1\}, \quad 1 \leq i, j, k \leq n, \\
 & \pi \in \mathbb{R}^n, \epsilon \in \mathbb{R}^n.
 \end{aligned}$$

$L(\pi, \epsilon)$ is a concave function as minimum of affine-linear functions. For finding its maximum a subgradient method can be used: Start with $\pi^r := \epsilon^r := 0$, use a greedy algorithm for evaluating $L(\pi^r, \epsilon^r)$ and let x_{ijk}^r be the corresponding optimal solution. Define $v_{i_0}^r := |\{x_{i_0,j,k}^r | x_{i_0,j,k} = 1\}| - 1$ for all $i_0 = 1, 2, \dots, n$ and $w_{j_0}^r := |\{x_{i,j_0,k}^r | x_{i,j_0,k} = 1\}| - 1$ for all $j_0 = 1, 2, \dots, n$. If $v^r = w^r = (0, 0, \dots, 0)$, then the maximum is reached. Otherwise π and ϵ are updated with a suitable step length λ ,

$$\pi^{r+1} := \pi^r + \lambda_r w^r, \quad \epsilon^{r+1} := \epsilon^r + \lambda_r v^r$$

and the next iteration is started. For details see Burkard and Rudolf (1993).

Other techniques

In connection with the application of semidefinite programming to combinatorial optimization problems, various other techniques from convex optimization were applied to discrete optimization problems. One of the most interesting approaches is due to Brixius and Anstreicher (2001) and concerns quadratic assignment problems (QAPs). Quadratic assignment problems which are very important for the practice, but notoriously hard to solve, can be stated as trace minimization problems of the form

$$\min \text{tr} (AXB + C)X^t,$$

where A, B and C are given $n \times n$ matrices and X is an $n \times n$ permutation matrix. First, one can relax the permutation matrix to an orthogonal matrix with row and column sum equal to 1. Then one can separate the linear and the quadratic term in the objective function. Brixius and Anstreicher interpret the relaxed problem in terms of semidefinite programming and evaluate a new bound which requires the solution of a convex quadratic program. This is performed via an interior point algorithm. The solution of the quadratic program allows to fix variables for the studied QAP and leads to very good computational results.

5. Convex configurations and combinatorial optimization problems

Many combinatorial optimization problems become easier to solve, if the input stems from convex sets. For example, the following fact about the **planar travelling salesman problem** (TSP), i.e., a TSP where the distances between the cities are given by (Euclidean) distances in the plane, is well known. Assume that the cities lie on the boundary of a convex set in the plane. Then an optimal solution is obtained by passing through the cities in clockwise or counterclockwise order on the

boundary. The reason for this is that in an optimal Hamiltonian cycle in the Euclidean plane the edges of the cycle never cross due to the quadrilateral inequality. Due to convexity every other solution than the clockwise or anticlockwise tour would have some crossing edges. It can be tested in $O(n \log n)$ time whether n given points in the plane lie on the boundary of a convex set, see e.g. Preparata and Shamos (1988). Their cyclic order can be found within the same time. If a distance matrix for a planar TSP is given, it can be tested in $O(n^2)$ time whether this is a distance matrix of vertices of a convex polygon or not (see Hotje's procedure in Burkard (1990)). Thus the case of a planar TSP whose cities are vertices of a convex polygon can easily be recognized and solved even though the planar TSP is \mathcal{NP} -hard in general (see Papadimitriou (1977)).

The same arguments as above apply, if the distances between cities are measured in the l_1 -norm and the cities are vertices of a rectilinearly convex set in the plane. A region R is called *rectilinearly convex* if every horizontal or vertical line intersects R in an interval.

The distance matrix $C = (c_{ij})$ of a planar TSP whose vertices lie on the boundary of a convex polygon has a special structure. The matrix fulfills the so-called *Kalmanson conditions*

$$c_{ij} + c_{kl} \leq c_{ik} + c_{jl} \quad \text{for all } 1 \leq i < j < k < l \leq n, \quad (2.7)$$

$$c_{il} + c_{jk} \leq c_{ik} + c_{jl} \quad \text{for all } 1 \leq i < j < k < l \leq n. \quad (2.8)$$

Kalmanson (1975) showed that a TSP whose distance matrix fulfills these Kalmanson conditions has the tour $\langle 1, 2, \dots, n-1, n \rangle$ as optimal solution, i.e. the travelling salesperson starts in city 1, goes then to city 2, and so on until she or he returns from city n to city 1. The definition of the Kalmanson property depends on a suited numbering of the rows and columns (i.e. of the cities) of the distance matrix. If after a renumbering of the rows and columns a matrix becomes a Kalmanson matrix, we speak of a *permuted Kalmanson matrix*. Permuted Kalmanson matrices can be recognised in $O(n^2)$ time by a method due to Christopher, Farach and Trick (see Christopher et al. (1996) and Burkard et al. (1998)). Permuted Kalmanson matrices are also interesting in connection with the so-called *master tour problem*. A *master tour* π for a set V of cities fulfills the following property: for every $V' \subseteq V$ an optimum travelling salesman tour for V' is obtained by removing from π the cities that are not in V' . Deĭneko, Rudolf and Woeginger (see Deĭneko et al. (1998)) showed that the master tour property holds if and only if the distance matrix is a permuted Kalmanson matrix.

Now let us turn to the **minimum spanning tree problem** (MST). Let a finite undirected and connected graph $G = (V, E)$ with vertex set V and edge set E be given. Every edge e has a positive length $c(e)$. (MST) asks for a spanning tree $T = (V, E_T)$, $E_T \subseteq E$, of G such that

$$\sum_{e \in E_T} c(e)$$

is minimum. If n points in the plane are given, the graph G is given by the complete complete graph K_n of these points and the edge lengths are given as (Euclidean) distances between the points. We have

Theorem 5.1 *A minimum spanning tree for n points in the plane can be computed in $O(n \log n)$ time. If the points lie on the boundary of a convex set and are given in cyclic order, the MST problem can be solved in $O(n)$ time.*

The idea behind this theorem is (see e.g. Mehlhorn (1984b)) that a minimum spanning tree of the given points contains only edges of the Delauney triangulation of these points. According to Aggarwal et al. (1989) the Delaunay triangulation of vertices of a convex polygon can be computed in $O(n)$ time. The Delaunay triangulation leads to a planar graph. Mehlhorn (1984a) showed that the MST in a planar graph can be solved in $O(n)$ time.

Similar results hold for the maximum spanning tree problem (see Monma et al. (1990)).

Now let us turn to the **Steiner tree problem** (STP) which has many applications in network design or VLSI design. The Steiner tree problem asks for the shortest connection of k given points, called *terminals* where it is allowed to introduce additional points, the so-called *Steiner points*. For example, if the terminals are the vertices of an equilateral triangle, then the center of gravity of the triangle is introduced as Steiner point. The connection of the Steiner point with each of the terminals yields the shortest Steiner tree of the given points. The length of a Steiner tree is again measured as sum of the lengths of all edges in the tree. The Steiner tree problem is **\mathcal{NP} -hard** in general (see Garey et al. (1977)). A Steiner tree problem is called *Euclidean*, if the terminals lie in the plane and all distances are measured in the Euclidean metric. For Euclidean Steiner tree problems, Provan (1988) showed the following result.

Theorem 5.2 *If the terminals of a Euclidean Steiner tree problem lie on the boundary of a convex set in the plane, then there exists a fully polynomial approximation scheme, i.e., there is an algorithm which constructs for any fixed $\epsilon > 0$ a Steiner tree T of length $l(T)$ such that*

$$l(T) \leq (1 + \epsilon) \cdot \text{Opt}$$

where Opt is the optimum value of the problem under consideration and where the running time of the algorithm is polynomial in n and $1/\epsilon$.

An even better result can be shown if the distances between vertices are measured in the l_1 -norm. This problem plays a special role in VLSI design where the connections between points use only horizontal or vertical lines of a grid. Provan (1988) showed

Theorem 5.3 *If the n terminal nodes of a Steiner tree problem lie on the boundary of a rectilinearly convex set and the distances between vertices are measured in the l_1 -norm, then the Steiner tree problem can be solved in $O(n^6)$ time.*

Now let us turn to **matching and assignment problems** in the plane. Let $2n$ points on the boundary of a convex set in the plane be given. We consider the complete graph K_{2n} whose vertices are these points and whose edge lengths are the Euclidean distances between the points. The weight of a matching M equals the sum of all edge lengths of M . Marcotte and Suri (1991) showed that a minimum weight matching in this K_{2n} can be found in $O(n \log n)$ time. Moreover, they showed that a maximum weight matching can be found in linear time.

Next we color n vertices of this K_{2n} red and n vertices blue and we allow edges only between vertices of different color. This gives rise to a matching problem in a bipartite graph (*assignment problem*). Marcotte and Suri (1991) showed also that the assignment problem defined above can be solved in $O(n \log n)$ time. Moreover, the verification of a minimum matching can be performed in $O(n \cdot \alpha(n))$ steps, where $\alpha(n)$ is the very slow growing inverse Ackermann function.

6. Conclusion

In the previous sections we outlined some of the important roles convexity plays in theory and practice of integer programming. But there are many other areas in discrete optimization, where (generalized) convexity is crucial. Let me just mention location problems, combinatorial optimization problems involving Monge arrays and submodular functions.

In **location theory** one wants to place one or more service centers such that the customers are served best. Classical location models lead to convex objective functions. The convexity of these functions is exploited in fast algorithms for solving these problems. For example, the simple form of Goldman's algorithm (see Goldman (1971)) for finding

the 1-median in a tree is mainly due to the convexity of the corresponding objective function.

Secondly, I would like to mention **Monge arrays**. A real $m \times n$ matrix $C = (c_{ij})$ is called *Monge matrix*, if

$$c_{ij} + c_{rs} \leq c_{is} + c_{rj} \quad \text{for all } 1 \leq i < r \leq m, 1 \leq j < s \leq n. \quad (2.9)$$

Many combinatorial optimization problems turn out to be easier to solve, if the problems are related to a Monge matrix. For example, if the cost coefficients of a transportation problem fulfill the Monge property (2.9), then the transportation problem can be solved in a greedy way by the north west corner rule. Or, if the distances of a travelling salesman problem fulfill the Monge property, then the TSP can be solved in linear time. A survey on Monge properties and combinatorial optimization can be found in Burkard, Klinz and Rudolf (see Burkard et al. (1996)). Monge matrices are closely related to **submodular functions**. A set function $f : 2^V \rightarrow \mathbb{R}$ is called submodular, if

$$f(X) + f(Y) \geq f(X \cup Y) + f(X \cap Y) \quad \text{for all } X, Y \subseteq V.$$

Submodular functions exhibit many features similar to convex functions and they play among others an important role in combinatorial optimization problems involving matroids. For details, the reader is referred to the pioneering work of Murota (1998).

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