One-Sided Monge TSP Is NP-Hard

Vladimir Deineko[1] and Alexander Tiskin[2]

[1] Warwick Business School, The University of Warwick, Coventry CV47AL, UK
V.Deineko@warwick.ac.uk
[2] Dept. of Computer Science, The University of Warwick, Coventry CV47AL, UK
tiskin@dcs.warwick.ac.uk

Abstract. The Travelling Salesman Problem (TSP) is a classical NP-hard optimisation problem. There exist, however, special cases of the TSP that can be solved in polynomial time. Many of the well-known TSP special cases have been characterized by imposing special *four-point conditions* on the underlying distance matrix. Probably the most famous of these special cases is the TSP on a *Monge matrix*, which is known to be polynomially solvable (as are some other generally NP-hard problems restricted to this class of matrices). By relaxing the four-point conditions corresponding to Monge matrices in different ways, one can define other interesting special cases of the TSP, some of which turn out to be polynomially solvable, and some NP-hard. However, the complexity status of one such relaxation, which we call *one-sided Monge TSP* (also known as the TSP on a *relaxed Supnick matrix*), has remained unresolved. In this note, we show that this version of the TSP problem is NP-hard. This completes the full classification of all possible four-point conditions for symmetric TSP.

1 Introduction

The travelling salesman problem (TSP) is a well-known problem of combinatorial optimisation. In the symmetric TSP, given a symmetric $n \times n$ distance matrix $C = (c_{ij})$, one looks for a cyclic permutation τ of the set $\{1, 2, \ldots, n\}$ that minimises the function $c(\tau) = \sum_{i=1}^{n} c_{i,\tau(i)}$. The value $c(\tau)$ is called the *length* of the permutation τ. We will in the following refer to the items in τ as *points*.

The TSP is an NP-hard problem [10]. There exist, however, special cases of the TSP that can be solved in polynomial time. For a survey of efficiently solvable cases of the TSP, see [3, 11, 14]. Many of the well-known TSP special cases result from imposing special conditions on the underlying distance matrix. Probably the most famous of these special cases is the TSP on a Monge matrix. For a number of well-known NP-hard problems, including TSP, restriction to Monge matrices reduces the complexity to polynomial (see survey [5]).

We give below two equivalent definitions of a Monge matrix. In what follows, we will always assume that the matrices under considerations are symmetric.

Definition 1. *An* $n \times n$ *matrix* $C = (c_{ij})$ *is a* Monge matrix*, if it satisfies the* Monge conditions*:*

$$c_{ij} + c_{i'j'} \le c_{ij'} + c_{i'j} \qquad 1 \le i < i' \le n \qquad 1 \le j < j' \le n \tag{1}$$

M. Gavrilova et al. (Eds.): ICCSA 2006, LNCS 3982, pp. 793–801, 2006.

Definition 2. *An $n \times n$ matrix $C = (c_{ij})$ is a* Monge matrix*, if*

$$c_{ij} + c_{i+1,j+1} \leq c_{i,j+1} + c_{i+1,j} \qquad 1 \leq i \leq n-1 \qquad 1 \leq j \leq n-1$$

It can be easily seen that the above two definitions define the same set of matrices. The difference arises when we begin to generalize the problem by relaxing the conditions imposed on the matrix. For example, the diagonal entries are not involved in the calculation of the TSP objective function, so one can define c_{ii} $(i = 1, \ldots, n)$ arbitrarily without affecting the solution of the TSP. If we relax Definition 1 by excluding the inequalities containing the diagonal entries, then the structural properties of the matrix remain essentially unchanged. In fact, given a matrix (c_{ij}) with indeterminate diagonal elements, which satisfies inequalities (1) for $i \neq j$, $i \neq j'$, $i' \neq j$, $i' \neq j'$, it is always possible to define diagonal elements c_{ii} so that the resulting matrix is a Monge matrix (see Proposition 2.13 in [3]). This relaxation of symmetric Monge matrices is known as *Supnick matrices*. Alternatively, Supnick matrices are defined as matrices satisfying conditions

$$c_{ij} + c_{j+1,l} \leq c_{i,j+1} + c_{jl} \leq c_{il} + c_{j,j+1} \qquad 1 \leq i < j < j+1 < l \leq n$$

Supnick [21] has shown that an optimal TSP tour on such matrices is given by $\sigma_{\text{Smin}} = \langle 1, 3, 5, 7, 9, \ldots, 8, 6, 4, 2, 1 \rangle$.

The well-known classes of Demidenko [9] and Van der Veen [23] matrices, as well as a class of matrices investigated in [4], can also be viewed as relaxations of Definition 1. A symmetric $n \times n$ matrix $C = (c_{ij})$ is a *Demidenko matrix*, if

$$c_{ij} + c_{i'j'} \leq c_{ij'} + c_{i'j} \qquad 1 \leq j < i < i' < j' \leq n$$

and a *Van der Veen matrix*, if

$$c_{ij} + c_{i'j'} \leq c_{ij'} + c_{i'j} \qquad 1 \leq i < j < i' < j' \leq n$$

The TSP on these classes of matrices is polynomially solvable: an optimal tour can be found among the so-called pyramidal tours in $O(n^2)$ time (see e.g. [3]).

Now, instead of Definition 1, we consider Definition 2, and relax it by excluding inequalities involving the diagonal elements.

Definition 3. *An $n \times n$ matrix $C = (c_{ij})$ is a* relaxed Supnick matrix*, if*

$$c_{ij} + c_{i+1,j+1} \leq c_{i,j+1} + c_{i+1,j} \qquad 1 \leq i < j-1 \leq n-2$$

In contrast with the Supnick relaxation of Definition 1, the relaxation from Definition 2 to Definition 3 may have a very significant effect on the matrix structure. While the Supnick conditions constrain all pairs of matrix elements excluding the main diagonal, the relaxed Supnick conditions only constrain pairs of matrix elements that are on the same side of the main diagonal (i.e. are either both above, or, by symmetry, both below the main diagonal). In calling such matrices "relaxed Supnick", we follow the terminology of [6]; otherwise, they could naturally be called *one-sided Monge matrices*.

Table 1. Classification of four-point conditions (adapted from [6])

	$\mathcal{A} \le \mathcal{B}$	$\mathcal{A} \ge \mathcal{B}$	$\mathcal{A} \le \mathcal{C}$	$\mathcal{A} \ge \mathcal{C}$	$\mathcal{B} \le \mathcal{C}$	$\mathcal{B} \ge \mathcal{C}$
$\mathcal{A} \le \mathcal{B}$	$O(n^2)$ [9]	$O(1)$ [11, 24]	$O(n^2)$ [9, 23]	$O(1)$ [15]	$O(1)$ [21]	$O(1)$ [15]
$\mathcal{A} \ge \mathcal{B}$		NP-hard [8]	$O(1)$ [6]	NP-hard [8, 22]	$O(n)$ [15, 19, 20]	$O(n)$ [21]
$\mathcal{A} \le \mathcal{C}$			$O(n^2)$ [23]	$O(1)$ [11, 24]	$O(1)$ [6]	$O(1)$ [6]
$\mathcal{A} \ge \mathcal{C}$				NP-hard [22]	$O(1)$ [6]	$O(1)$ [6]
$\mathcal{B} \le \mathcal{C}$					**NP-hard (new)**	$O(n^2)$ [16, 6]
$\mathcal{B} \ge \mathcal{C}$						$O(n^4)$ [6]

Given a relaxed Supnick matrix with indeterminate diagonal elements, it is not always possible to define the diagonal elements so that the resulting matrix is a Monge matrix, as the following example shows:

$$\begin{pmatrix} \times & 1 & 0 & 1 & 5 \\ 1 & \times & 0 & 0 & 3 \\ 0 & 0 & \times & 1 & 1 \\ 1 & 0 & 1 & \times & 0 \\ 5 & 3 & 1 & 0 & \times \end{pmatrix}$$

Computational complexity of the TSP on the described matrix classes, and other classes of similar type, can be studied systematically by considering *four-point conditions.* Let i, j, k, l be four points with $1 \le i < j < k < l \le n$. A symmetric distance matrix for these points contains six entries above the main diagonal, which correspond to the six edges connecting these points. It is possible to form three pairs of non-incident edges: $\{(i,j),(k,l)\}$, $\{(i,k),(j,l)\}$, $\{(i,l),(j,k)\}$. We denote the combined lengths of these pairs by $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, respectively:

$$\mathcal{A} = c_{ij} + c_{kl} \qquad \mathcal{B} = c_{ik} + c_{jl} \qquad \mathcal{C} = c_{il} + c_{jk}$$

A *four-point condition* is an inequality relation among the values $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, which has to be satisfied for all possible choices of indices i, j, k, l with $1 \le i < j < k < l \le n$. Using this notation, Supnick matrices are defined as matrices satisfying conditions $\mathcal{A} \le \mathcal{B}$, $\mathcal{B} \le \mathcal{C}$, Demidenko matrices as matrices satisfying condition $\mathcal{A} \le \mathcal{B}$, Van der Veen matrices as matrices satisfying conditions $\mathcal{A} \le \mathcal{C}$, and relaxed Supnick matrices as matrices satisfying condition $\mathcal{B} \le \mathcal{C}$. Nearly all possible four-point conditions and their pairwise combinations have been classified in [6] (see Table 1) according to the polynomial solvability or NP-hardness of the arising TSP. The only gap that has remained in this classification is the TSP on relaxed Supnick matrices. In the following section we show that this problem is NP-hard, thus making a final point in the classification of four-point conditions for symmetric TSP.

2 The TSP on Relaxed Supnick Matrices

As before, we assume that all the considered matrices are symmetric.

Theorem 1. *The TSP on a relaxed Supnick matrix is NP-hard.*

Proof. The proof is by reduction from the Hamiltonian cycle problem in grid graphs [13]. We follow the definitions from [13]. Let G^∞ be the infinite graph, whose vertex set consists of all the integer points in the plane, and in which two vertices are connected, if and only if the Euclidean distance between them is equal to one. A grid graph is a finite node-induced subgraph of G^∞. It is shown in [13] that the Hamiltonian cycle problem in a grid graph is NP-hard.

Given a grid graph on n nodes, we will construct an $n \times n$ relaxed Supnick matrix with non-negative entries, such that there exists a Hamiltonian cycle in the graph, if and only if the optimal TSP tour on the corresponding relaxed Supnick matrix has zero length.

Consider an arbitrary grid graph G on n nodes. We may assume that G is connected (otherwise, it is guaranteed not to contain a Hamiltonian cycle). First, we embed G in a square grid of size $m \times m$, where m is sufficiently large. Due to the connectivity of G, we may assume $m \leq n$. We then extend the square grid to a parallelogram grid, and number the nodes as shown in Figure 1. The upper-left point in the square grid will be numbered m, the two neighbouring points $2m-1$ and $2m$, etc. Let $N = m^2 + m(m-1)$ be the number of points in the parallelogram grid. We then construct an $N \times N$ relaxed Supnick matrix $C^{(N)}$ as follows. For two neighbouring nodes i, j, $i < j$, in the parallelogram grid, we define $c_{ij}^{(N)} = 0$, if both i and j belong to G. Notice that the numbering of points in the parallelogram grid is chosen so that all these entries are placed on two

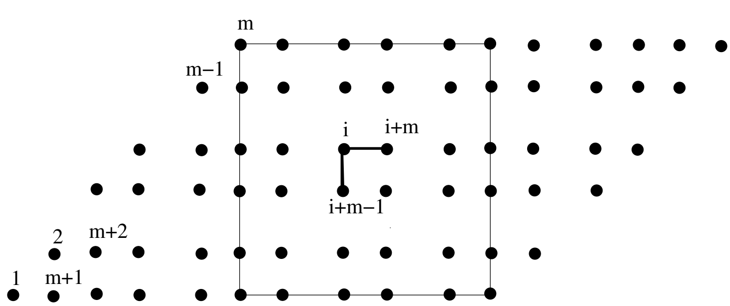

Fig. 1. Embedding a square grid in a parallelogram

adjacent diagonals: $j-i \in \{m-1, m\}$. For all the remaining entries in these two diagonals, we define $c_{ij}^{(N)} = 1$. Further, we define $c_{1j}^{(N)} = 1$ for $j = 1, \ldots, m-1$, and $c_{iN}^{(N)} = 1$ for $i = N-m+2, \ldots, N$.

Notice that so far there exist no four defined entries in the upper triangular part of $C^{(N)}$ that would form an inequality from Definition 2, therefore none of the one-sided Monge conditions are violated. We keep filling in the upper triangular part of the matrix, respecting the Monge conditions. We define

$$\begin{aligned} c_{ij}^{(N)} &= \max\{c_{i-1,j}^{(N)} + c_{i,j+1}^{(N)} - c_{i-1,j+1}^{(N)}, 1\} \qquad j-i = 1, \ldots, m-2 \\ c_{ij}^{(N)} &= \max\{c_{i,j-1}^{(N)} + c_{i+1,j}^{(N)} - c_{i+1,j-1}^{(N)}, 1\} \qquad j-i = m+1, \ldots, N-1 \end{aligned}$$

for each element, as soon as the right-hand side of this element's definition has itself been defined. We always have $c_{ij}^{(N)} \geq 1$ unless i, j are neighbouring nodes in G, and the one-sided Monge condition is always preserved. The lower triangular part of $C^{(N)}$ is defined by symmetry. The construction of the relaxed Supnick matrix $C^{(N)}$ is now completed.

It can be easily checked that, given a relaxed Supnick matrix C, the symmetric matrix obtained by deleting a row and a corresponding column from C is still relaxed Supnick. By induction, the same is true after deleting an arbitrary subset of rows and corresponding columns. By deleting all rows and columns from $C^{(N)}$, except those indexed by points in the original grid graph G, we obtain an $n \times n$ relaxed Supnick matrix $C^{(n)}$. Clearly, there exists a Hamiltonian cycle in graph G, if and only if the optimal TSP tour on the matrix $C^{(n)}$ has zero length.

In order to prove that the above problem reduction is polynomial, it only remains to show that the growth of individual matrix elements is appropriately bounded. Observe that each new element created in $C^{(N)}$ cannot exceed the largest existing element by more than a factor of 2. Consequently, the largest element in $C^{(N)}$ has value[1] at most $2^{N^2} \leq 2^{(2m^2)^2} = 2^{4m^4} \leq 2^{4n^4}$, and can therefore be represented by a number of bits that is polynomial in n. Hence, even if our computational model does not support unit-cost arithmetic on arbitrarily long integers, arithmetic operations on matrix elements can be emulated bitwise in polynomial time. The reduction is completed. □

3 Relaxed Supnick Matrices and Exponential Neighbourhoods

The research into polynomially solvable cases for intractable problems is partly motivated by the hope to identify new approaches to constructing general-purpose heuristics. One of the by-products of this research are exponential neighbourhoods (see surveys [1, 7]), which are being intensively studied in relation to

[1] Much tighter estimates are possible. However, this crude bound is sufficient for our proof.

local search algorithms. Among new families of exponential neighbourhoods presented in [6], there is a neighbourhood of *strongly balanced permutations*, which are related to relaxed Supnick matrices. To justify the introduction of this neighbourhood, the authors of [6] considered a special subclass of relaxed Supnick matrices.

Definition 4. *A relaxed Supnick matrix* $C = (c_{ij})$ *is* strong*, if*

$$c_{tp} - c_{tz} - c_{kp} \leq c_{sy} - c_{ky} - c_{sz} \qquad t < k < s < p < z < y$$

It can be shown that the inequalities in Definition 4 are equivalent to the system of inequalities

$$c_{ij} - c_{i,j+1} - c_{i+1,j} \leq c_{j-1,N} - c_{i+1,N} - c_{j-1,j} \qquad i < j$$

and can therefore be checked in time $O(n^2)$.

As shown in [6], an optimal tour for the TSP on a strong relaxed Supnick matrix can be found in the set of so-called strongly balanced tours. We first give some preliminary definitions. An index $i \in \{1, \ldots, n\}$ is a *peak* of a permutation τ, if $i > \max\{\tau^{-1}(i), \tau(i)\}$, and a *valley*, if $i < \min\{\tau^{-1}(i), \tau(i)\}$. An index which is neither peak nor valley is called *intermediate*. Informally, in a strongly balanced tour, all intermediate nodes are "evenly spread" on the slopes between peaks and valleys. We now give a formal definition.

We will consider partially constructed tours on the sets of indices $\{1, 2, 3, \ldots, m-1, m\}$ with $m = 1, 2, \ldots, n$. For a fixed m, a partially constructed tour will consist of a set of finite index sequences; indices from each sequence are placed in the tour in consecutive positions. We refer to each of these sequences $\langle i_1, \ldots, j_1 \rangle$ as *fragment* $[i_1, j_1]$, stressing that i_1 is the initial, and j_1 is the final element in the corresponding sequence. Notice that it is not necessary for a fragment $[i_1, j_1]$ with $i_1 < j_1$ to contain, for example, $i_1 + 1$. For a one-element fragment we use the notation $[i, i]$. For example, if we start with a tour where 1 and 2 are two valleys, we can represent this initial tour by two fragments $[1, 1]$ and $[2, 2]$. Mutual placement of fragments is not fixed, i.e. they can be permuted. The fragments can also be reversed, i.e. fragment $[i, j]$ can be replaced by fragment $[j, i]$.

Definition 5 ([6]). *A tour is* strongly balanced*, if it can be constructed as follows. Start with an initial tour* $[1, 1]$ *and repeat the folowing step for* $m = 2, \ldots, n-1$*:*

> *Given a partial tour on the set of indices* $\{1, \ldots, m-1\}$*, the tour is represented by fragments* $[i_1, j_2]$*,* $[i_2, j_2]$*, …,* $[i_p, j_p]$*. Let*
>
> $$i_{min1} = \min\{i_1, j_1, i_2, j_2, \ldots, i_p, j_p\}$$
> $$i_{min2} = \min\{i_1, j_1, i_2, j_2, \ldots, i_p, j_p\} \setminus \{i_{min1}\}$$
> $$i_{min3} = \min\{i_1, j_1, i_2, j_2, \ldots, i_p, j_p\} \setminus \{i_{min1}, i_{min2}\}$$

Add index m to the partial tour by choosing one of the options below:

- *m is placed as a new valley; this creates a new fragment $[m, m]$;*
- *m is placed as an intermediate index adjacent to i_{min1}; fragment $[i_{min1}, s]$ in the partially constructed tour is replaced by the new fragment $[m, s]$;*
- *m is placed as a new peak merging two fragments; m is adjacent to i_{min1} and to either i_{min2} or to i_{min3}. In the first case, the fragments $[i_{min1}, j]$ and $[i_{min2}, s]$ are merged into $[j, s]$; in the second case, the fragments $[i_{min1}, i_{min2}]$ and $[i_{min3}, s]$ are merged into $[i_{min2}, s]$.*

The final node n can only be added to a partial tour consisting of one fragment.

As an example, the reader can check that the tour

$$\langle 1, 4, 10, 6, 2, 7, 12, 9, 3, 8, 11, 5, 1\rangle$$

with the intermediate nodes 4, 5, 6, 7, 8, 9 evenly spread on the slopes, is a strongly balanced tour.

Proposition 1 ([6]). *An optimal tour for the TSP with a strong reduced Supnick matrix can be found in the set of strongly balanced tours.*

Despite the relatively simple structure of strongly balanced tours, the problem of finding an optimal tour in this set appears to be difficult. To avoid the difficulties, the authors of [6] have further restricted the special class of matrices, and identified a subset of strongly balanced tours, where an optimal tour can be found in polynomial time.

Proposition 2 ([6]). *Consider strongly balanced tours for which the maximum number of fragments in the construction of Definition 5 is bounded by a constant. An optimal tour in this special subset of strongly balanced tours can be found in polynomial time.*

We can now explain why the problem of finding an optimal strongly balanced tour is difficult for an unbounded number of fragments.

Theorem 2. *The TSP on a strong relaxed Supnick matrix is NP-hard.*

Proof. The proof is similar to the proof of Theorem 1. We define the initial values $c_{ij}^{(N)}$ in two adjacent diagonals: $j - i \in \{m - 1, m\}$ exactly as before. The only difference is in the way we define the remaining entries. To ensure that the matrix is a strong relaxed Supnick matrix, it is sufficient to define

$$\begin{aligned} c_{ij}^{(N)} = \max\{ & c_{j,i-1}^{(N)} + c_{j+1,i-1}^{(N)} - c_{j+1,i-1}^{(N)}, \\ & c_{j,i-1}^{(N)} + c_{i-1,i}^{(N)} + c_{j+1,N}^{(N)} - c_{i-1,N}^{(N)} - c_{j+1,i-1}^{(N)}, 1\} \qquad j < j + m < i \end{aligned}$$

and

$$\begin{aligned} c_{ip}^{(N)} = \max\{ & c_{i-1,p}^{(N)} + c_{i,i+1}^{(N)} - c_{i-1,p+1}^{(N)}, \\ & c_{i-1,p}^{(N)} + c_{p,p+1}^{(N)} + c_{iN}^{(N)} - c_{pN}^{(N)} - c_{i-1,p+1}^{(N)}, 1\} \qquad i < p < i + m \end{aligned}$$

The rest of the proof is identical to the proof of Theorem 1, with the only difference that the largest element in $C^{(N)}$ has now value at most $3^{N^2} \leq 3^{4n^4}$. □

Corollary 1. *The problem of finding an optimal strongly balanced tour is NP-hard.*

The two statements above justify the introduction of a special subset of strongly balanced tours (Proposition 2), where the optimal permutation can be found in polynomial time.

4 Conclusion

We have shown that the TSP on a relaxed Supnick matrix, which can also be viewed as a one-sided Monge matrix, is NP-hard. This completes the classification of all possible four-point conditions for symmetric TSP [6]. Our results justify imposing further restrictions on relaxed Supnick matrices in order to identify new polynomially solvable cases of the TSP.

We hope that the constructions in this note, which reduce the TSP on a grid graph to the TSP with a special matrix, in combination with the special polynomially solvable case described in Proposition 2 (see [6] for further details), may lead to identifying special types of grid graphs, where the optimal tour can be found in polynomial time. Some polynomially solvable cases of the TSP on grid graphs have already been identified in [2, 18].

References

1. Ahuja, R.K., Ergun, Ö., Orlin, J.B., Punnen, A.: A survey of very large-scale neighborhood search techniques. Discrete Applied Mathematics, **123** (2002) 75–102
2. Arkin, E.M., Bemder, M.A., Demaine, E., Fekete, S.P., Mitchell, J.S., Sthia, S.: Optimal covering tours with turn costs. In Proc. 13th ACM–SIAM Symposium on Discrete Algorithms, SODA01 (2001) 138–147
3. Burkard, R.E., Deineko, V.G., van Dal, R., van der Veen, J.A.A., Woeginger, G.J.: Well-solvable special cases of the TSP: A survey. SIAM Review, 40, 3 (1998) 496–546
4. Burkard, R.E., Deineko, V.G.: On the traveling salesman problem with a relaxed Monge matrix. Information Processing Letters, 67 (1998) 231–237
5. Burkard, R.E., Klinz, B.,Rudolf, R.: Perspectives of Monge Properties in Optimization. Discrete Applied Mathematics, 70 (1996) 95–161
6. Deineko, V.G., Klinz, B., Woeginger, G.J.: Four point conditions for symmetric TSP and exponential neighbourhoods. In Proc. 17th ACM–SIAM Symposium on Discrete Algorithms, SODA 06 (2006) 544–553
7. Deineko, V.G., Woeginger, G.J.: A study of exponential neighborhoods for the travelling salesman problem and for the quadratic assignment problem. Mathematical Programming 87 (2000) 519–542
8. Deineko, V.G., Woeginger, G.J.: The maximum travelling salesman problem on symmetric Demidenko matrices. Discrete Applied Mathematics, 99 (2000) 413–425

9. Demidenko, V.M.: A special case of traveling salesman problems. Izv. Akad. Nauk. BSSR, Ser. Fiz.-mat. Nauk, 5 (1976) 28–32 (in Russian)
10. Garey, M.R., Johnson, D.S.: Computers and intractability. Freeman and Co., San Francisco (1979)
11. Gilmore, P.C., Lawler, E.L., Shmoys, D.B.: Well-solved special cases. Chapter 7 in [17] (1985) 207–249
12. Gutin, G., Punnen, A.P. (eds.): The travelling salesman problem and its variations. Kluwer Academic Publishers (2002)
13. Itai, A., Papadimitiou, C.H., Szwarcfiter, J.L.: Hamiltonian paths in grid graphs. SIAM Journal on Computing, 11, 4 (1982) 676–686
14. Kabadi, S.N.: Polynomially solvable cases of the TSP. Chapter 11 in [12] (2002) 489–583
15. Kalmanson, K.: Edgeconvex circuits and the traveling salesman problem. Canadian Journal of Mathematics, 27 (1975) 1000–1010
16. Lawler, E.L.: A solvable case of the traveling salesman problem. Mathematical Programming, 1 (1971) 267–269
17. Lawler, E.L., Lenstra, J.K., Rinnooy Kan, A.H.G., Shmoys, D.B.: The Traveling Salesman Problem. Wiley, Chichester (1985)
18. Lenhart, W., Umans, U.: Hamiltonian cycles in solid grid graphs. In Proceedings of 38 Annual Symposium on Foundations of Computer Science, FOCS '97 (1997) 496–596
19. Quintas, L.V., Supnick, F.: Extreme Hamiltonian Circuits: Resolution of the convex-odd case. Proc. Amer. Math. Soc., 15 (1964) 454–459
20. Quintas, L.V., Supnick, F.: Extreme Hamiltonian Circuits: Resolution of the convex-even case. Proc. Amer. Math. Soc., 16 (1965) 1058–1061
21. Supnick, F.: Extreme Hamiltonian lines. Annals of Math., 66 (1957) 179–201
22. Steiner, G., Xue, X.: The maximum traveling salesman problem on van der Veen matrices. Discrete Applied Mathematics, 146 (2005) 1–2
23. van der Veen, J.A.A.: A new class of pyramidally solvable symmetric traveling salesman problems. SIAM J. Discr. Math., 7 (1994) 585–592
24. Volodarskiy, Y.M., Gabovich, E.Y., Zacharin, A.Y.: A solvable case in discrete programming. Izv. Akad. Nauk. SSSR, Ser. Techn. Kibernetika, 1 (1976) 34–44 (in Russian). English translation in Engineering Cybernetics, 14 (1977) 23–32