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CONIC CHARACTERIZATION OF MONGE MATRICES

V. M. Demidenko

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A complete description is given to the linearity space of Monge cone matrices and of all its minimum faces. The description makes it possible to completely characterize Monge matrices. Possible applications of the results obtained are discussed.

Keywords: *combinatory optimization problems, polynomial-solvable special cases, Monge matrices, cones, systems of generatrices.*

INTRODUCTION

Revealing polynomial or strongly solvable cases is important in the analysis of NP-difficult discrete problems. As a rule, they are determined by the conditions imposed on the form of input data, whose realization guarantees either availability of polynomial solution algorithms or presentation of an optimum without additional calculations.

For optimization combinatory problems with linear objective functionals, whose input data are real matrices, the conditions of polynomial and strong solvability can be written as redundant homogeneous systems of linear inequalities [1–3] that specify cones of a special form in the corresponding space. Therefore, of theoretical and practical interest is to find subsystems, minimum in the number of inequalities, that describe solvability cones, and their minimum edges (extreme rays in case of cone tapering). For example, for some optimization problems on substitutions, the above systems of inequalities allow selecting sufficiently wide classes of adjacent vertices of polytopes [4], and systems of cones generatrices of strong solvability [5–7] allow constructing their various relaxations.

Randomly generated representatives of the cones of strong solvability can be applied in quality evaluation of heuristic and approximate algorithms and in testing exact methods of solution of optimization problems. Various relaxations of polytopes constructed based on the same cones allow us to use well-developed tools of linear programming for calculating lower estimates used in the “branches and bounds” methods.

PRELIMINARY INFORMATION

The so-called Monge conditions are the well-known conditions of polynomial, in particular, strong solvability for some optimization problems. They were first described in 1781 by the French mathematician G. Monge [8]. If we denote by $C=[c_{ij}]$ a real $m\times n$-matrix, then the Monge conditions can be written as

$$c_{ik}+c_{jl}-c_{il}-c_{jk}\le 0,\ \ 1\le i<j\le m,\ 1\le k<l\le n\,. \tag{1}$$

The matrices satisfying (1) are hereafter called Monge matrices. The interest recently shown in the Monge conditions [3] is determined by the fact that for some NP-difficult problems they guarantee availability of efficient solution algorithms, and for polynomial-solvable problems, reduce the labor input needed for their solution or strong solvability. For example, under the Monge conditions, the NP-difficult general problem on allocations [9] admits a significant contraction of the set of

Institute of Mathematics, National Academy of Sciences of Belarus, Minsk, Belarus, *demidenko@im.bas-net.by*.
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admissible solutions [10, 11], and its special cases —: the ordinary problem of allocations and the traveling salesman problem — become strongly [11, 12] and polynomial [1] solvable, respectively. In this case, the latter is reduced to searching for optimum on a set of pyramidal cycles [1, 13]. When the Monge conditions are realized, it is solved in time $O(n)$ [14, 15], though time $O(n^2)$ is required for its solution in the general case [16, 17]. For similar conditions, the problem of graph partitioning [18] and the transport problem [19] are solved in a linear time, and the problem about the minimum weighed pair matching with additional condition for the weight matrix form (differences of two adjacent elements in each row do not decrease) requires time $O(n \log_2 m)$ for its solution [20]. A detailed review on application of the Monge conditions to discrete optimization problems is presented in [3].

Let $\mathbf{R}$ be the field of real numbers, $\mathbf{R}^{m\times n}$ be the space of real $m\times n$-matrices, and $\mathbf{E}$ be the base of space $\mathbf{R}^{m\times n}$, derived by 0, 1-matrices E_{ij}, $i = 1,\ldots, m$, $j = 1,\ldots, n$, each having a unique nonzero element at the intersection of the ith row and jth column. The equalities below immediately follow from the definition of the matrices E_{ij}:

$$(E_{ij}, E_{ij}) = 1, \ \ (E_{ij}, E_{kl}) = 0, \ \ i \neq k \ \text{or} \ j \neq l. \qquad (2)$$

Using $\mathbf{E}$, let us define for arbitrary i, j, k, l, where $1 \le i < j \le m$, $1 \le k < l \le n$, the matrices

$$A_{ijkl} = E_{ik} + E_{jl} - E_{il} - E_{jk} \qquad (3)$$

and denote by $\mathbf{A}$ the set of all such matrices. For notation convenience, put $A_{ijkl} = A_{ik}$ for $j = i+1$ and $l = k+1$.

If we denote further by $(A, B) = \sum_{p=1}^{m} \sum_{q=1}^{n} a_{pq} b_{pq}$ a scalar product of an arbitrary pair of matrices $A = [a_{ij}]$, $B = [b_{ij}]$ from space $\mathbf{R}^{m\times n}$, and by X the matrix of variables x_{pq}, $p = 1,\ldots, m$, $q = 1,\ldots, n$, then the Monge conditions (1) can be written as a homogeneous system of linear inequalities slu $\mathbf{A} = \{(A, X) \le 0 \mid A \in \mathbf{A}\}$. It is obvious that the set of solutions of the system slu $\mathbf{A}$ (the set of Monge matrices) forms in $\mathbf{R}^{m\times n}$ some cone $\mathbf{K}_{\text{MON}}$, which is called the Monge cone. Hereinafter, we use the following notation: $\mathbf{L}_{\text{MON}}$ is the linearity space of the cone $\mathbf{K}_{\text{MON}}$ (the maximum subspace of space $\mathbf{R}^{m\times n}$ contained in $\mathbf{K}_{\text{MON}}$), $\mathbf{L}^{\perp}_{\text{MON}}$ is the orthogonal complement of $\mathbf{L}_{\text{MON}}$ in $\mathbf{R}^{m\times n}$, and $\mathbf{K}^{\perp}_{\text{MON}}$ is the orthogonal projection of $\mathbf{K}_{\text{MON}}$ onto $\mathbf{L}^{\perp}_{\text{MON}}$, i.e., $\mathbf{K}^{\perp}_{\text{MON}} = \mathbf{L}^{\perp}_{\text{MON}} \cap \mathbf{K}_{\text{MON}}$.

Minimum polyhedral descriptions of $\mathbf{K}_{\text{MON}}$ and $\mathbf{K}^{\perp}_{\text{MON}}$ are obtained here, i.e., irredundant linear systems are separated out, which define in the matrix space $\mathbf{R}^{m\times n}$ the cones mentioned. This allows us to write out explicitly their minimum edges and finally to give a complete conic characterization of Monge matrices. Possible applications of the results obtained are also discussed. It should be noted that additive characterization of Monge matrices with non-negative elements was obtained earlier [21], which differs from the proposed one in that it does not give a description of minimum edges of the cones $\mathbf{K}_{\text{MON}}$ and $\mathbf{K}^{\perp}_{\text{MON}}$.

IRREDUNDANT POLYHEDRAL DESCRIPTION OF THE MONGE CONE

In the present section, the subsystem (minimum in the number of inequalities) of the system slu $\mathbf{A}$ that defines the Monge cone $\mathbf{K}_{\text{MON}}$ is described. The subsystem makes it possible to obtain irredundant descriptions of spaces $\mathbf{L}_{\text{MON}}$ and $\mathbf{L}^{\perp}_{\text{MON}}$ and the cone $\mathbf{K}^{\perp}_{\text{MON}}$. Let us select in the set of matrices $\mathbf{A}$ a subset $\mathbf{A}_1 = \{A_{ij} \in \mathbf{A} \mid i = 1, \ldots, m-1, \ j = 1, \ldots, n-1\}$, which determines the subsystem of the system slu $\mathbf{A}$ of the form slu $\mathbf{A}_1 = \{(A_{ij}, X) \le 0 \mid A_{ij} \in \mathbf{A}_1\}$.

LEMMA 1. The system of linear inequalities slu $\mathbf{A}$ follows from its subsystem slu $\mathbf{A}_1$.

Proof. By virtue of the Minkowski–Farkas lemma about corollary inequalities [22, 119], to prove Lemma 1, it will suffice to prove that matrices from $\mathbf{A}$ can be presented as linear combinations (with non-negative coefficients) of matrices from $\mathbf{A}_1$. Hereafter, such combinations of matrices are called conic. Let A_{ijkl} be an arbitrary matrix from $\mathbf{A}$. Let us show that conic representation of the form

$$A_{ijkl} = \sum_{r=i}^{j-1} \sum_{s=k}^{l-1} A_{rs}, \ \ 1 \le i < j \le m, \ \ 1 \le k < l \le n \qquad (4)$$

holds for it. Let us first verify the following equalities:

$$A_{rjkl} = A_{rr+1kl} + A_{r+1jkl}, \quad r = i, \ldots, j-2. \tag{5}$$

Substituting decompositions on the base $\mathbf{E}$ of the matrices from (3) into the right-hand sides of (5), we obtain

$$A_{rr+1kl} + A_{r+1jkl} = E_{rk} + E_{r+1l} - E_{rl} - E_{r+1k} + E_{r+1k} + E_{jl} - E_{r+1l} - E_{jk} = A_{rjkl}.$$

Thus, validity of (5) is proved. Adding with respect to $r = i, \ldots, j-2$ equalities from (5) and collecting terms, we obtain for the matrix A_{ijkl} the conic representation

$$A_{ijkl} = \sum_{r=i}^{j-1} A_{rr+1kl}, \quad 1 \le i < j \le m, \; 1 \le k < l \le n. \tag{6}$$

Similarly, we prove the equalities $A_{rr+1sl} = A_{rs} + A_{rr+1s+1l}$ for all $s = k, \ldots, l-2$, whose term-by-term addition with collecting terms yields $A_{rr+1kl} = \sum_{s=k}^{l-1} A_{rs}$, where $1 \le r < r+1 \le m$, $1 \le k < l \le n$. Substitution of the right-hand side of the obtained equality instead of the matrix A_{rr+1kl} into (6) proves (4). Lemma 1 is proved.

LEMMA 2. The matrices from $\mathbf{A}_1$ are linearly independent in $\mathbf{R}^{m \times n}$.

Proof. For determinacy, $m < n$ put and write the linear combination of matrices $\mathbf{A}_1$ as the sum

$$L(\mathbf{A}_1) = \sum_{i=1}^{m-1} \sum_{j=i}^{n-1} \alpha_{ij} A_{ij} + \sum_{i=1}^{m-2} \sum_{j=i+1}^{m-1} \beta_{ji} A_{ji}, \; \alpha_{ij}, \; \beta_{ji} \in \mathbf{R}. \tag{7}$$

To prove Lemma 2, it will suffice to show that all the coefficients in (7) are equal to zero in the case $L(\mathbf{A}_1) = O$, where O is zero matrix, i.e., the equalities are fulfilled

$$\begin{aligned} &\alpha_{ij} = 0, \; i = 1, \ldots, m-1, \; j = i, \ldots, n-1, \\ &\beta_{ji} = 0, \; i = 1, \ldots, m-2, \; j = i+1, \ldots, m-1. \end{aligned} \tag{8}$$

Let $m < n$. Then, replacing in (7) the matrices A_{ij}, A_{ji} by their expansions in the base $\mathbf{E}$

$$A_{ij} = E_{ij} + E_{i+1j+1} - E_{ij+1} - E_{i+1j}, \; A_{ji} = E_{ji} + E_{j+1i+1} - E_{ji+1} - E_{j+1i}$$

and grouping in (7) all the coefficients for each matrix from $\mathbf{E}$, we obtain

$$\begin{aligned} L(\mathbf{A}_1) = &\sum_{i=2}^{m-1} \sum_{j=i+1}^{n-1} (\alpha_{ij} + \alpha_{i-1j-1} - \alpha_{ij-1} - \alpha_{i-1j}) E_{ij} + \sum_{j=2}^{n-1} (\alpha_{1j} - \alpha_{1j-1}) E_{1j} - \alpha_{1n-1} E_{1n} \\ &+ \sum_{i=2}^{m-1} (\alpha_{i-1n-1} - \alpha_{in-1}) E_{in} + \sum_{j=m}^{n-1} (\alpha_{m-1j-1} - \alpha_{m-1j}) E_{mj} + \alpha_{m-1n-1} E_{mn} \\ &+ \sum_{i=2}^{m-3} \sum_{j=i+2}^{m-1} (\beta_{ji} + \beta_{j-1i-1} - \beta_{ji-1} - \beta_{j-1i}) E_{ji} + \sum_{j=3}^{m-1} (\beta_{j1} - \beta_{j-11}) E_{j1} - \beta_{m-11} E_{m1} \\ &+ \sum_{i=2}^{m-2} (\beta_{m-1i-1} - \beta_{m-1i}) E_{mi} + (\beta_{21} - \alpha_{11}) E_{21} \\ &+ \sum_{i=2}^{m-2} (\beta_{i+1i} + \beta_{ii-1} - \beta_{i+1i-1} - \alpha_{ii}) E_{i+1i} + \beta_{m-1m-2} E_{mm-1} + \alpha_{11} E_{11} \\ &+ \sum_{i=2}^{m-1} (\alpha_{ii} + \alpha_{i-1i-1} - \alpha_{i-1i} - \beta_{ii-1}) E_{ii}. \end{aligned} \tag{9}$$

If now $L(\mathbf{A}_1)=O$, then by virtue of the linear independence of matrices from $\mathbf{E}$, all the coefficients in (9) must be equal to zero, which yields, in particular, the following equalities:

$$\alpha_{ij}+\alpha_{i-1j-1}-\alpha_{ij-1}-\alpha_{i-1j}=0,\ i=2,\dots,m-1,\ \ j=i+1,\dots,n-1, \tag{10}$$

$$\alpha_{1n-1}=0,\ \alpha_{1j}-\alpha_{1j-1}=0,\ \ j=2,\dots,n-1, \tag{11}$$

$$\alpha_{i-1n-1}-\alpha_{in-1}=0,\ \ i=2,\ \dots,m-1. \tag{12}$$

The validity of $\alpha_{ij}-\alpha_{ij-1}=0$ for all $i=2,\dots,m-1,\ j=i+1,\dots,n-1$, follows from (10) and (11), which, in turn, results in fulfillment of the chains of the equalities

$$\alpha_{ii}=\alpha_{ii+1}=\dots=\alpha_{in-2}=\alpha_{in-1},\ \ i=2,\dots,m-1. \tag{13}$$

By analogy, $\alpha_{1n-1}=0$ and (12) yields $0=\alpha_{1n-1}=\alpha_{2n-1}=\dots=\alpha_{m-2n-1}=\alpha_{m-1n-1}$, whence with regard for (13) we obtain the equalities

$$\alpha_{ij}=0,\quad i=2,\dots,m-1,\quad j=i,\dots,n-1. \tag{14}$$

Finally, the equality $\alpha_{11}=\alpha_{12}=\dots=\alpha_{1n-2}=\alpha_{1n-1}=0$ follows from (11), wherefrom, with regard for (14), we have $\alpha_{ij}=0$ for all $i=1,\dots,m-1,\ j=i,\dots,n-1$.

For $L(\mathbf{A}_1)=O$, by virtue of the linear independence of matrices from $\mathbf{E}$, coefficients from (9) must be zero, i.e., the following equalities must hold:

$$\beta_{ji}+\beta_{j-1i-1}-\beta_{ji-1}-\beta_{j-1i}=0,\ \ i=2,\dots,m-3,\ j=i+2,\dots,m-1, \tag{15}$$

$$\beta_{m-11}=0,\ \beta_{m-1i-1}-\beta_{m-1i}=0,\ \ i=2,\ \dots,m-2, \tag{16}$$

$$\beta_{j1}-\beta_{j-11}=0,\ \ j=3,\ \dots,m-1. \tag{17}$$

The equalities $\beta_{ji}-\beta_{j-1i}=0$ for all $i=2,\dots,m-3,\ j=i+2,\ \dots,m-1$, follow from (15) and (17), whence we obtain the chains of the equalities

$$\beta_{i+1i}=\beta_{i+2i}=\dots=\beta_{m-2i}=\beta_{m-1i},\ \ i=2,\ \dots,m-3. \tag{18}$$

The equalities $0=\beta_{m-11}=\beta_{m-12}=\dots=\beta_{m-1m-3}=\beta_{m-1m-2}$ follow from (16), whence, with regard for (18), the equalities $\beta_{ji}=0$ follow for all $i=2,\dots,m-2,\ j=i+1,\dots,m-1$. Finally, it follows from $\beta_{m-11}=0$ and (17) that $0=\beta_{m-11}=\beta_{m-21}=\dots=\beta_{31}=\beta_{21}$. Thus, (8) is proved for the case $m<n$. The case $m>n$ can be proved similarly. If $m=n$, then, replacement in (9) of all items related to the coefficients α_{ij} by the expression

$$\sum_{i=2}^{m-2}\ \sum_{j=i+1}^{m-1}(\alpha_{ij}+\alpha_{i-1j-1}-\alpha_{ij-1}-\alpha_{i-1j})E_{ij}$$

$$+\sum_{j=2}^{m-1}(\alpha_{1j}-\alpha_{1j-1})E_{1j}-\alpha_{1m-1}E_{1m}+\sum_{i=2}^{m-1}(\alpha_{i-1m-1}-\alpha_{im-1})E_{im}+\alpha_{m-1m-1}E_{mm}$$

and reasoning similar to the case $m<n$ yield (8) for $m=n$. Lemma 2 is proved.

LEMMA 3. The following equalities are true: $\operatorname{rank}(\operatorname{slu}\mathbf{A}_1)=(m-1)(n-1)$, $\dim\mathbf{L}_{\mathrm{MON}}=m+n-1$.

Proof. Since by virtue of Lemma 2 the matrices from $\mathbf{A}_1$ are linearly independent in $\mathbf{R}^{m\times n}$, $|\mathbf{A}_1|=(m-1)(n-1)$, and the number of variables in the system $\operatorname{slu}\mathbf{A}_1$ is $m\times n$, we have $\operatorname{rank}(\operatorname{slu}\mathbf{A}_1)=(m-1)(n-1)$, and $\dim\mathbf{L}_{\mathrm{MON}}=m\times n-\operatorname{rank}(\operatorname{slu}\mathbf{A}_1)=m+n-1$. Lemma 3 is proved.

THEOREM 1. The system of inequalities $\operatorname{slu}\mathbf{A}_1$ whose set of solutions coincides with $\mathbf{K}_{\mathrm{MON}}$, is a subsystem from $\operatorname{slu}\mathbf{A}$, minimum in the number of inequalities.

Proof. Let there be a subset $\mathbf{A}_2=\{A_1,\dots,A_s\}$ in the set $\mathbf{A}$ such that $|\mathbf{A}_2|<(m-1)(n-1)$ and the system $\operatorname{slu}\mathbf{A}$ is a corollary of its subsystem $\operatorname{slu}\mathbf{A}_2=\{(A_i,X)\le 0\,|\,i=1,\dots,s\}$. Denote by $\mathbf{L}_2$ space of solutions of the homogeneous system of linear equations slg $\mathbf{A}_2$, which can be obtained from slu $\mathbf{A}_2$ by replacing of all signs $\le$ with $=$. By the Minkowskii–Farkas lemma [22, p. 119], the equalities $(A_{ijkl}, X) = \sum_{r=1}^{s} \lambda_r (A_r, X)$ should be fulfilled for arbitrary matrices A_{ijkl} from $\mathbf{A}$ and X from $\mathbf{R}^{m\times n}$, where $0 \le \lambda_r \in \mathbf{R}$. This yields $(A_{ijkl}, X) = 0$ for all X from $\mathbf{L}_2$, $1 \le i < j \le m$, $1 \le k < l \le n$, whence the inclusion $\mathbf{L}_2 \subseteq \mathbf{L}_{\mathrm{MON}}$ is true and, therefore, $\dim \mathbf{L}_2 \le \dim \mathbf{L}_{\mathrm{MON}} = m + n - 1$. Let $\mathbf{L}_2^{\perp}$ be orthogonal complement of $\mathbf{L}_2$ in $\mathbf{R}^{m\times n}$, then $\dim \mathbf{L}_2^{\perp} = \mathrm{rank}\,(\mathrm{slg}\,\mathbf{A}_2) \le |\mathbf{A}_2| < (m-1)(n-1)$. Further, since $\dim \mathbf{R}^{m\times n} = \dim \mathbf{L}_2 + \dim \mathbf{L}_2^{\perp}$, we have

$$\dim \mathbf{L}_2 = \dim \mathbf{R}^{m\times n} - \dim \mathbf{L}_2^{\perp} = mn - \mathrm{rank}\,(\mathrm{slg}\,\mathbf{A}_2)$$

$$\ge mn - |\mathbf{A}_2| > mn - (m-1)(n-1) = m + n - 1$$

that contradicts the established inequality $\dim \mathbf{L}_2 \le \dim \mathbf{L}_{\mathrm{MON}}$. The inconsistency obtained and Lemma 1 prove the validity of Theorem 1.

LEMMA 4. The set

$$\mathbf{E}_1 = \left\{ E_i' = \sum_{s=1}^{n} E_{is},\ E_j'' = \sum_{r=1}^{m} E_{rj} \ |\, i = 1, \ldots, m,\ j = 1, \ldots, n-1 \right\}$$

is the base of the space $\mathbf{L}_{\mathrm{MON}}$.

Proof. According to [22, p. 155] and Theorem 1, space $\mathbf{L}_{\mathrm{MON}}$ coincides with the set of solutions of the system of equations slg $\mathbf{A}_1 = \{(A_{kl}, X) = 0 \,|\, A_{kl} \in \mathbf{A}_1\}$. By virtue of $|\mathbf{E}_1| = m + n - 1$ and Lemma 3, to prove Lemma 4 it will suffice to prove that the matrices from $\mathbf{E}_1$ are solutions of the system slg $\mathbf{A}_1$ and are linearly independent in $\mathbf{R}^{m\times n}$. Using the expansion of the matrices A_{kl} and E_i' in the base $\mathbf{E}$, where $1 \le k \le m-1$, $1 \le l \le n-1$, $1 \le i \le m$, we obtain

$$(A_{kl}, E_i') = \left(E_{kl} + E_{k+1\,l+1} - E_{kl+1} - E_{k+1\,l},\ \sum_{s=1}^{n} E_{is} \right)$$
$$= \sum_{s=1}^{n} (E_{kl}, E_{is}) + \sum_{s=1}^{n} (E_{k+1l+1}, E_{is}) - \sum_{s=1}^{n} (E_{kl+1}, E_{is}) - \sum_{s=1}^{n} (E_{k+1l}, E_{is}), \tag{19}$$

whence $(A_{kl}, E_i') = 0$ for $i \ne k,\ k+1$ since in this case all items in (19) are equal to zero by virtue of (2). If $i = k$, then by virtue of (2) we have

$$\sum_{s=1}^{n} (E_{kl}, E_{ks}) = (E_{kl}, E_{kl}) = 1, \quad \sum_{s=1}^{n} (E_{kl+1}, E_{ks}) = (E_{kl+1}, E_{kl+1}) = 1,$$

$$\sum_{s=1}^{n} (E_{k+1l+1},\ E_{ks}) = 0, \quad \sum_{s=1}^{n} (E_{k+1l}, E_{ks}) = 0.$$

Substituting the sums with the values found into (19), we obtain $(A_{kl}, E_k') = 0$. If $i = k+1$, by virtue of (2), the following equalities must hold:

$$\sum_{s=1}^{n} (E_{k+1l+1},\ E_{k+1s}) = (E_{k+1l+1}, E_{k+1l+1}) = 1,$$

$$\sum_{s=1}^{n} (E_{k+1l}, E_{k+1s}) = (E_{k+1l}, E_{k+1l}) = 1,$$

$$\sum_{s=1}^{n} (E_{kl}, E_{k+1s}) = 0, \quad \sum_{s=1}^{n} (E_{kl+1}, E_{k+1s}) = 0.$$

These equalities and (19) yield $(A_{kl}, E'_{k+1}) = 0$. Similarly, for matrices A_{kl} and E''_j we obtain

$$(A_{kl}, E_j'') = \sum_{s=1}^{m} (E_{kl}, E_{sj}) + \sum_{s=1}^{m} (E_{k+1l+1}, E_{sj})$$

$$-\sum_{s=1}^{m} (E_{kl+1}, E_{sj}) - \sum_{s=1}^{m} (E_{k+1l}, E_{sj}), \tag{20}$$

where $1 \le k \le m-1$, $1 \le l$, $j \le n-1$. Therefore, by virtue of (2), the scalar products (A_{kl}, E''_j) are equal to zero for $j \ne l$, $l+1$. Further, it follows from (20) and (2) that for $j = l$ and $j = l+1$, respective scalar products have the form

$$(A_{kl}, E''_l) = (E_{kl}, E_{kl}) - (E_{k+1l}, E_{k+1l}) = 1 - 1 = 0,$$

$$(A_{kl}, E''_{l+1}) = (E_{k+1l+1}, E_{k+1l+1}) - (E_{kl+1}, E_{kl+1}) = 1 - 1 = 0.$$

Thus, it is proved that the matrices E'_i and E''_j are solutions of the system of equations slg $\mathbf{A}_1$ for all $i = 1, \ldots, m$, $j = 1, \ldots, n-1$.

To prove the linear independence of the matrices from $\mathbf{E}_1$, let us consider a linear combination

$$L(\mathbf{E}_1) = \sum_{i=1}^{m} \lambda_i E'_i + \sum_{j=1}^{n-1} \mu_j E''_j,$$

where $\lambda_i, \mu_j \in \mathbf{R}$, and show that when $L(\mathbf{E}_1) = O$, all the coefficients λ_i and μ_j are equal to zero. Substituting the expansions of the matrices E'_i and E''_j in the base $\mathbf{E}$ into $L(\mathbf{E}_1)$, where $1 \le i \le m$, $1 \le j \le n-1$, and performing simple transformations, we obtain

$$L(\mathbf{E}_1) = \sum_{i=1}^{m} \lambda_i \left(\sum_{j=1}^{n} E_{ij} \right) + \sum_{j=1}^{n-1} \mu_j \left(\sum_{i=1}^{m} E_{ij} \right) = \sum_{i=1}^{m} \sum_{j=1}^{n-1} \lambda_i E_{ij} + \sum_{i=1}^{m} \lambda_i E_{in}$$

$$+ \sum_{i=1}^{m} \sum_{j=1}^{n-1} \mu_j E_{ij} = \sum_{i=1}^{m} \sum_{j=1}^{n-1} (\lambda_i + \mu_j) E_{ij} + \sum_{i=1}^{m} \lambda_i E_{in}.$$

If $L(\mathbf{E}_1) = O$, then by virtue of the linear independence of the matrices from $\mathbf{E}$ the equalities $\lambda_i + \mu_j = 0$, $\lambda_i = 0$, $i = 1, \ldots, m$, $j = 1, \ldots, n-1$, must hold whence all the coefficients of the linear combination $L(\mathbf{E}_1)$ must be equal to zero, i.e., the matrices from $\mathbf{E}_1$ are linearly independent. Thus, Lemma 4 is proved.

LEMMA 5. Space $\mathbf{L}^{\perp}_{\text{MON}}$ coincides with the set of solutions of the system of equations

$$\text{slg}\, \mathbf{E}_1 = \{(E'_i, X) = 0,\ (E''_j, X) = 0 \mid i = 1, \ldots, m,\ j = 1, \ldots, n-1\}.$$

Proof. Validity of Lemma 5 immediately follows from the definition of the subspace $\mathbf{L}^{\perp}_{\text{MON}}$ as an orthogonal complement of $\mathbf{L}_{\text{MON}}$ in $\mathbf{R}^{m \times n}$.

Denote by slu $\mathbf{K}^{\perp}_{\text{MON}}$ an irredundant system whose set of solutions coincides with $\mathbf{K}^{\perp}_{\text{MON}}$. According to [23, p. 158], such a system exists and is unique to within multiplying inequalities by positive scalars from $\mathbf{R}$. The following theorem describes its form.

THEOREM 2. The equality slu $\mathbf{K}^{\perp}_{\text{MON}} =$ slg $\mathbf{E}_1 \cup$ slu $\mathbf{A}_1$ is true.

Proof. By the definition from [23, p. 155], the cone $\mathbf{K}^{\perp}_{\text{MON}}$ is the intersection of $\mathbf{L}^{\perp}_{\text{MON}}$ and $\mathbf{K}_{\text{MON}}$. Therefore, the union of systems slg $\mathbf{E}_1$ and slu $\mathbf{A}_1$ that describe $\mathbf{L}^{\perp}_{\text{MON}}$ and $\mathbf{K}_{\text{MON}}$, respectively, determines the cone $\mathbf{K}^{\perp}_{\text{MON}}$. Since by virtue of Lemmas 2, 4, and 5 the systems slg $\mathbf{E}_1$ and slu $\mathbf{A}_1$ are irredundant, the system slg $\mathbf{E}_1 \cup$ slu $\mathbf{A}_1$ is also irredundant and, therefore, it coincides with slu $\mathbf{K}^{\perp}_{\text{MON}}$. Thus, Theorem 2 is proved.

By virtue of Theorem 2 and Lemmas 3 and 5, the rank of the system slu $\mathbf{K}^{\perp}_{\text{MON}}$ = slg $\mathbf{E}_1 \cup$ slu $\mathbf{A}_1$, which determines $\mathbf{K}^{\perp}_{\text{MON}}$, coincides with the dimension of space $\mathbf{R}^{m\times n}$. Thus, the linearity space of the cone $\mathbf{K}^{\perp}_{\text{MON}}$, determined by the system of equations slu $\mathbf{K}^{\perp}_{\text{MON}}$ = slg $\mathbf{E}_1 \cup$ slg $\mathbf{A}_1$, is a zero subspace of space $\mathbf{R}^{m\times n}$. Therefore, $\mathbf{K}^{\perp}_{\text{MON}}$ is an acute cone, and its minimum edges are extreme rays. According to [22, p. 162], the directrices of these rays are determined uniquely to within positive factors from $\mathbf{R}$ and coincide with the set of fundamental solutions of the system slu $\mathbf{K}^{\perp}_{\text{MON}}$. Let us describe this set. Let $\mathbf{A}_{ij} = \mathbf{A}_1 \setminus A_{ij}$, slu $\mathbf{A}_{ij} = \{(A, x) \le 0 \mid A \in \mathbf{A}_{ij}\}$, and slg $\mathbf{A}_{ij}$ be the system of linear equations obtained from slu $\mathbf{A}_{ij}$ by replacing all signs $\le$ with $=$, where $i = 1, \dots, m-1$, $j = 1, \dots, n-1$.

LEMMA 6. Let X_{ij} be the solution of the system of linear equations slg $\mathbf{E}_1 \cup$ slg $\mathbf{A}_{ij}$ such that $(A_{ij}, X_{ij}) < 0$, where $1 \le i \le m-1$, $1 \le j \le n-1$. Then the set of matrices

$$\{X_{ij} \mid i = 1, \dots, m-1,\ j = 1, \dots, n-1\}$$

coincides with the set of all fundamental solutions of the system slu $\mathbf{K}^{\perp}_{\text{MON}}$.

Proof. According to [22, p. 162], the solution of the system slu $\mathbf{K}^{\perp}_{\text{MON}}$ is fundamental if it turns all the inequalities of some subsystem from slu $\mathbf{K}^{\perp}_{\text{MON}}$ with the rank rank(slu $\mathbf{K}^{\perp}_{\text{MON}}$) $- 1$ into equalities. Since |slu $\mathbf{K}^{\perp}_{\text{MON}}$| = rank(slu $\mathbf{K}^{\perp}_{\text{MON}}$) $= mn$, any subsystem of the rank $mn - 1$ can be obtained by deleting from slu $\mathbf{K}^{\perp}_{\text{MON}}$ either the inequality $(A_{ij}, X) \le 0$, $1 \le i \le m-1$, $1 \le j \le n-1$, or the equations $(E'_i, X) = 0$, $1 \le i \le m$, or the equation $(E''_j, X) = 0$, $1 \le j \le n-1$. The subsystems obtained from slu $\mathbf{K}^{\perp}_{\text{MON}}$ by deleting one equation cannot generate fundamental solutions of slu $\mathbf{K}^{\perp}_{\text{MON}}$ since their nonzero solutions (the necessary fundamentality condition) must not turn to zero the left-hand side of the deleted equation (otherwise, we have the solution slg $\mathbf{K}^{\perp}_{\text{MON}}$ and, therefore, it is zero). But such solutions are not fundamental solutions of slu $\mathbf{K}^{\perp}_{\text{MON}}$. Since an arbitrary fundamental solution $X_{ij} \ne 0$ generated, for example, by the system slg $\mathbf{E}_1 \cup \mathbf{A}_{ij}$, must turn all of its inequalities into equalities, it is a solution of the system of equations slg $\mathbf{E}_1 \cup$ slg $\mathbf{A}_{ij}$ and simultaneously of the system slu $\mathbf{K}^{\perp}_{\text{MON}}$, i.e., the strict inequality $(A_{ij}, X_{ij}) < 0$ must hold for it (otherwise it will be a solution of slg $\mathbf{K}^{\perp}_{\text{MON}}$ for $(A_{ij}, X_{ij}) = 0$, whence $X_{ij} = 0$, and for $(A_{ij}, X_{ij}) > 0$ $X_{ij} \notin \mathbf{K}^{\perp}_{\text{MON}}$). To complete the proof of Lemma 6, it remains to verify that there exists the solution X_{ij} of an arbitrary systems of equations slg $\mathbf{E}_1 \cup$ slg $\mathbf{A}_{ij}$, $1 \le i \le m-1$, $1 \le j \le n-1$, such that $(A_{ij}, X_{ij}) < 0$. Since the rank of each homogeneous system of equations slg $\mathbf{E}_1 \cup$ slg $\mathbf{A}_{ij}$, $1 \le i \le m-1$, $1 \le j \le n-1$, is equal to $mn - 1$, the set of its solutions is a one-dimensional subspace $\mathbf{L}_1$ in $\mathbf{R}^{m\times n}$, in which there always exists a matrix $X_{ij} \ne O$ for which the strict inequality $(A_{ij}, X_{ij}) < 0$ holds. Indeed, $(A_{ij}, X_{ij}) < 0$ for all nonzero matrices from $\mathbf{L}_1$ (otherwise X_{ij} is the solution of slg $\mathbf{K}^{\perp}_{\text{MON}}$ whence $X_{ij} = O$). Let us select in $\mathbf{L}_1$ any matrix $X_{ij} \ne O$. If $(A_{ij}, X_{ij}) < 0$, then X_{ij} is the required fundamental solution. If $(A_{ij}, X_{ij}) > 0$, then $-X_{ij}$ from $\mathbf{L}_1$ is the required fundamental solution since $(A_{ij}, -X_{ij}) = -(A_{ij}, X_{ij}) < 0$. Lemma 6 is proved.

Let us introduce the set of matrices

$$\mathbf{U} = \{U_{ij} = [u^{(i,j)}_{pq}] \in \mathbf{R}^{m\times n} \mid i = 1, \dots, m-1, j = 1, \dots, n-1\}$$

whose elements are determined as follows:

$$u^{(i,j)}_{pq} = \begin{cases} -(m-i)(n-j) \text{ for all } 1 \le p \le i, 1 \le q \le j, \\ (m-i)j \text{ for all } 1 \le p \le i,\ j+1 \le q \le n, \\ (n-j)i \text{ for all } i+1 \le p \le m,\ 1 \le q \le j, \\ -ij \text{ for all } i+1 \le p \le m,\ j+1 \le q \le n. \end{cases} \qquad (21)$$

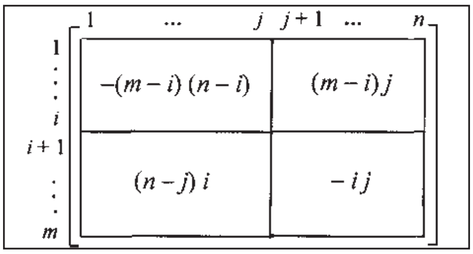

Fig. 1

It follows from (21) that each matrix U_{ij}, $1 \le i \le m-1$, $1 \le j \le n-1$, has a cell-like structure. Namely, the *i*th row and the *j*th column divide the matrix U_{ij} into four cells with the elements having the same values. Figure 1 schematically shows the matrix U_{ij}, $1 \le i \le m-1$, $1 \le j \le n-1$, with the value of matrix elements written inside each rectangle depicting the corresponding cell of the matrix U_{ij}.

THEOREM 3. An arbitrary system of directing extreme rays of the cone $\mathbf{K}_{\text{MON}}^{\perp}$ coincides to within positive factors from $\mathbf{R}$ with the set of matrices $\mathbf{U}$.

Proof. By virtue of Lemma 6, to prove Theorem 3 it will suffice to prove that the matrix U_{ij}, $1 \le i \le m-1$, $1 \le j \le n-1$, is a solution of the system of equations slg $\mathbf{E}_1 \cup$slg $\mathbf{A}_{ij}$ and the strict inequality $(A_{ij}, U_{ij}) < 0$ holds for it. Let us first show that an arbitrary matrix $U_{ij} = [u_{pq}^{(i,j)}]$ is a solution of the system slg $\mathbf{E}_1$. Using the expansion of the matrix E'_p, $1 \le p \le m$, in the base $\mathbf{E}$, we have

$$(E'_p, U_{ij}) = \left(\sum_{r=1}^{n} E_{pr}, U_{ij} \right) = \sum_{r=1}^{n} (E_{pr}, U_{ij}) = \sum_{r=1}^{n} u_{pr}^{(i,j)} = \sum_{r=1}^{j} u_{pr}^{(i,j)} + \sum_{r=j+1}^{n} u_{pr}^{(i,j)}. \tag{22}$$

Substituting the values of corresponding elements of the matrix U_{ij} into formula (22), we obtain from (21)

$$(E'_p, U_{ij}) = -\sum_{r=1}^{j} (m-i)(n-j) + \sum_{r=j+1}^{n} (m-i)j = -j(m-i)(n-j) + (n-j)(m-i)j = 0$$

if $1 \le p \le i$, and $(E'_p, U_{ij}) = \sum_{r=1}^{j} (n-j)i - \sum_{r=j+1}^{n} ij = j(n-j)i - (n-j)ij = 0$ if $i+1 \le p \le m-1$. By analogy, we are convinced that for the matrices E''_q, $1 \le q \le n-1$ the following equalities hold:

$$(E''_q, U_{ij}) = \left(\sum_{r=1}^{m} E_{rq}, U_{ij} \right) = \sum_{r=1}^{m} (E_{rq}, U_{ij}) = \sum_{r=1}^{m} u_{rq}^{(i,j)} = \sum_{r=1}^{i} u_{rq}^{(i,j)} + \sum_{r=i+1}^{m} u_{rq}^{(i,j)}. \tag{23}$$

Substituting from (21) the values of the elements of the matrix U_{ij} into (23), we obtain

$$(E''_q, U_{ij}) = -\sum_{r=1}^{i} (m-i)(n-j) + \sum_{r=i+1}^{m} (n-j)i = -i(m-i)(n-j) + (m-i)(n-j)i = 0$$

if $1 \le q \le j$, and $(E''_q, U_{ij}) = \sum_{r=1}^{i} (m-i)j - \sum_{r=i+1}^{m} ij = i(m-i)j - (m-i)ij = 0$ if $j+1 \le q \le n-1$. Thus, we have proved that the matrices from $\mathbf{U}$ are solutions of the system slg $\mathbf{E}_1$.

Let us now show that U_{ij} is a solution of the system of equations slg $\mathbf{A}_{ij}$ such that $(A_{ij}, U_{ij}) < 0$, where $1 \le i \le m-1$, $1 \le j \le n-1$. Let A_{pq} be a matrix from $\mathbf{A}_1$ such that $p \ne i$ or $q \ne j$. Then, using the expansion of A_{pq} in the base $\mathbf{E}$, we have

$$(A_{pq}, U_{ij}) = (E_{pq}, U_{ij}) + (E_{p+1q+1}, U_{ij}) - (E_{pq+1}, U_{ij}) - (E_{p+1q}, U_{ij}) = u_{pq}^{(i,j)} + u_{p+1q+1}^{(i,j)} - u_{pq+1}^{(i,j)} - u_{p+1q}^{(i,j)}.$$

Substituting the values of the elements of the matrix U_{ij} into the right-hand side of the equality from (21) and performing simple transformations, we are convinced that the following relations hold for $p \neq i$ or $q \neq j$:

$$(A_{pq}, U_{ij}) = \begin{cases} -(m-i)(n-j)-(m-i)(n-j)+(m-i)(n-j)+(m-i)(n-j)=0, \\ \qquad \text{if } 1 \le p \le i-1,\ 1 \le q \le j-1, \\ -(m-i)(n-j)+(m-i)j-(m-i)j+(m-i)(n-j)=0, \\ \qquad \text{if } 1 \le p \le i-1,\ q=j, \\ (m-i)j+(m-i)j-(m-i)j-(m-i)j=0, \\ \qquad \text{if } 1 \le p \le i-1,\ j+1 \le q \le n-1, \\ -(m-i)(n-j)+(n-j)i+(m-i)(n-j)-(n-j)i=0, \\ \qquad \text{if } p=i,\ 1 \le q \le j-1, \\ (m-i)j-ij-(m-i)j+ij=0, \text{ if } p=i, j+1 \le q \le n-1, \\ (n-j)i+(n-j)i-(n-j)i-(n-j)i=0, \\ \qquad \text{if } i+1 \le p \le m-1,\ 1 \le q \le j-1, \\ (n-j)i-ij+ij-(n-j)i=0, \text{ if } i+1 \le p \le m-1,\ q=j, \\ -ij-ij+ij+ij=0, \text{ if } i+1 \le p \le m-1,\ j+1 \le q \le n-1. \end{cases}$$

It follows from the above relations that U_{ij} is the solution of the system slg $\mathbf{A}_{ij}$ for all $1 \le i \le m-1$, $1 \le j \le n-1$. At last, for $p=i$ and $q=j$, substituting from (21) the values of the elements of the matrix U_{ij} into expression for (A_{pq}, U_{ij}) and performing simple transforms, we obtain the inequality

$$(A_{ij}, U_{ij}) = -(m-i)(n-j)-ij-(m-i)j-(n-j)i = -mn < 0.$$

Thus, by virtue of Lemma 6, the matrices from $\mathbf{U}$ form the set of fundamental solutions of the system slu $\mathbf{K}_{\text{MON}}^{\perp}$ and, therefore, they determine, to within multiplication by positive scalars from $\mathbf{R}$, the system of directing extreme rays of the cone $\mathbf{K}_{\text{MON}}^{\perp}$. Theorem 3 is proved.

It is noted in [23, p. 155] that an arbitrary nonempty polyhedron can be represented as a direct sum of its linearity space and its intersection with an orthogonal complement for this space. Therefore, the representation $\mathbf{K}_{\text{MON}} = \mathbf{L}_{\text{MON}} + \mathbf{K}_{\text{MON}}^{\perp}$ takes place for $\mathbf{K}_{\text{MON}}$, which, with regard of Theorem 3, makes it possible to give a complete conic characterization of Monge matrices.

THEOREM 4. For an arbitrary Monge's matrix C, there exist in $\mathbf{R}$ the numbers α_p and β_q, $1 \le p \le m$, $1 \le q \le n-1$, and non-negative numbers μ_{ij}, $1 \le i \le m-1$, $1 \le j \le n-1$, such, that

$$C = \sum_{p=1}^{m} \alpha_p E'_p + \sum_{q=1}^{n-1} \beta_q E''_q + \sum_{i=1}^{m-1} \sum_{j=1}^{n-1} \mu_{ij} U_{ij}. \tag{24}$$

Representation (24) for Monge matrices allows us to describe the minimum edges of the Monge cone. The following theorem shows this.

THEOREM 5. The sets of matrices $\mathbf{L}_U = \{L + \lambda U \mid L \in \mathbf{L}_{\text{MON}},\ 0 < \lambda \in \mathbf{R}\}$ where $U \in \mathbf{U}$, and only they are minimum edges of the cone $\mathbf{K}_{\text{MON}}$.

CONCLUSIONS

Selection in slu $\mathbf{A}$ of an irredundant subsystem slu $\mathbf{A}_1$ that describes the Monge cone allows us to solve the problem of recognizing membership to this cone of matrices from $\mathbf{R}^{m \times n}$ in time $O((m-1)(n-1))$. Due to the conic characterization of Monge's matrices of the form (24), an arbitrary (random) matrix from $\mathbf{K}_{\text{MON}}$ can be constructed in time $O(m^2 n^2)$. The description of minimum edges of the Monge cone allows us to reduce the problem on recognizing enumerated Monge's matrices to a special linear programming problem. Similar recognition problems were considered in [24]. Similar conic characterizations of the Supnik [25] and Kalmanson [26] symmetrical matrices for which strong solvability of the travelling salesman symmetrical problem is guaranteed are obtained in [6, 7] .

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