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The Cone of Monge Matrices: Extremal Rays and Applications[1]

RÜDIGER RUDOLF AND GERHARD J. WOEGINGER

TU Graz, Institut für Mathematik B, Steyrergasse 30, 8010 Graz, Austria

Abstract: We present an additive characterization of Monge matrices based on the extremal rays of the cone of nonnegative Monge matrices. By using this characterization, a simple proof for an old result by Supnick (1957) on the traveling salesman problem on Monge matrices is derived.

Key Words: Combinatorial optimization, traveling salesman problem, Monge matrix, cone.

1 Introduction

An $m \times n$ matrix $C = (c_{ij})$ is called a *Monge matrix* if it satisfies the *Monge property*

$$c_{ij} + c_{rs} \leq c_{is} + c_{rj} \qquad \forall 1 \leq i < r \leq m \ , \qquad 1 \leq j < s \leq n \ .$$

This property dates back to G. Monge [5] and is also known as *concave quadrangle inequality* (cf. e.g. Yao [8]). Monge matrices play an essential role in combinatorial optimization problems: For example, the NP-hard *traveling salesman problem* (TSP) is solvable in linear time if the distance matrix is a Monge matrix (cf. Park [6]). For the *Hitchcock transportation problem*, the north-west corner rule yields the optimal solution, if the underlying cost-matrix is Monge (cf. Hoffman [2]). For further examples the reader is referred to the survey by Burkard, Klinz and Rudolf [1].

Although it is easy to see that the class of $m \times n$ Monge matrices forms a cone K in the vector space of the $m \times n$ matrices with real entries, the extremal rays of K have not been investigated till now.

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In this note we determine the extremal rays of the cone of nonnegative Monge matrices and thereby derive simple additive characterizations for Monge matrices and symmetric Monge matrices. Moreover, it is shown how to apply these characterizations to simplify optimality proofs for several combinatorial optimization problems on Monge matrices. Intuitively speaking, whenever the combinatorial structure of the optimum solution to some optimization problem is identical for all extremal rays, then this combinatorial structure carries over to the optimum solution for *all* Monge matrices. This idea leads to very simple proofs for two results obtained by Supnick [7] on special cases of the traveling salesman problem.

2 The Cone of Monge Matrices

Unless stated otherwise, all matrices in this section are $m \times n$ matrices with real entries. The following $n + m + 2(n-1)(m-1)$ Monge matrices form the extremal rays of the cone of the $m \times n$ Monge matrices with *nonnegative* entries. For $1 \le i \le m$ define matrices $H^{(i)} = (h_{pq}^{(i)})$, for $1 \le j \le n$ define matrices $V^{(j)} = (v_{pq}^{(j)})$, for $1 \le i \le m-1$ and $2 \le j \le n$ define matrices $R^{(ij)} = (r_{pq}^{(ij)})$, and for $2 \le i \le m$ and $1 \le j \le n-1$ define matrices $L^{(ij)} = (l_{pq}^{(ij)})$ by

$$h_{pq}^{(i)} := \begin{cases} 1 & p = i \\ 0 & \text{otherwise} \end{cases} \qquad \forall i = 1, \ldots, m$$

$$v_{pq}^{(j)} := \begin{cases} 1 & q = j \\ 0 & \text{otherwise} \end{cases} \qquad \forall j = 1, \ldots, n$$

$$r_{pq}^{(ij)} := \begin{cases} 1 & p \le i,\, q \ge j \\ 0 & \text{otherwise} \end{cases} \qquad \forall i = 1, \ldots, m-1\ , \qquad j = 2, \ldots, n$$

$$l_{pq}^{(ij)} := \begin{cases} 1 & p \ge i,\, q \le j \\ 0 & \text{otherwise} \end{cases} \qquad \forall i = 2, \ldots, m\ , \qquad j = 1, \ldots, n-1\ .$$

Let $\mathscr{H} = \{H^{(i)} | 1 \le i \le m\}$, $\mathscr{V} = \{V^{(j)} | 1 \le j \le n\}$, $\mathscr{L} = \{L^{(ij)} | 2 \le i \le m, 1 \le j \le n-1\}$ and $\mathscr{R} = \{R^{(ij)} | 1 \le i \le m-1, 2 \le j \le n\}$, and let $\mathscr{M}$ be the union of $\mathscr{H}$, $\mathscr{V}$, $\mathscr{R}$ and $\mathscr{L}$. It is straightforward to check that all matrices in $\mathscr{M}$ are Monge matrices.

Observation 2.1: Let C and D be $m \times n$ Monge matrices and let $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^n$. Then the following matrices are Monge matrices as well.

(i) *the transpose* C^T,
(ii) *the matrix* λC *for* $\lambda \geq 0$,
(iii) *the sun* $C + D$ *and*
(iv) *the matrix* $A = (a_{ij})$ *defined by* $a_{ij} = c_{ij} + u_i + v_j$. □

Observation 2.2: If an $m \times n$ *matrix* C *is* not *a Monge matrix, then there exist indices* $1 \leq p \leq m$ *and* $1 \leq q \leq n$ *with* $c_{pq} + c_{p+1,q+1} > c_{p,q+1} + c_{p+1,q}$. □

Observation 2.3: For an $m \times n$ *Monge matrix* C*, let* $j(i)$ *denote the column index of the minimum entry in row* i*,* $1 \leq i \leq m$ *(if the minimum occurs more than once in this row,* $j(i)$ *is the index of the leftmost occurrence). Then* $j(1) \leq j(2) \leq \cdots \leq j(m)$ *holds.* □

Matrices that fulfill the above condition on the row minimum are called *monotone* matrices. The proof of the following lemma will make extensive use of Observations 2.1, 2.2 and 2.3 without explicitly stating this every time.

Lemma 2.4: For every nonnegative $m \times n$ *Monge matrix* C*, there exist nonnegative numbers* κ_i*,* λ_j*,* μ_{ij} *and* ν_{ij} *such that*

$$C = \sum_{i=1}^{m} \kappa_i H^{(i)} + \sum_{j=1}^{n} \lambda_j V^{(j)} + \sum_{i=2}^{m} \sum_{j=1}^{n-1} \mu_{ij} L^{(ij)} + \sum_{i=1}^{m-1} \sum_{j=2}^{n} \nu_{ij} R^{(ij)} \ . \tag{1}$$

Proof: Suppose the contrary, i.e. that there exist nonegative Monge matrices which cannot be represented as a nonnegative linear combination of the matrices in $\mathscr{M}$ as described in (1). Let C be a counterexample with the maximum number of zero entries. Clearly, every row and column of C contains at least one zero entry (In case C had, say, a row i with only non-zero entries, let α denote the smallest value in row i. Then the matrix $C - \alpha H^{(i)}$ constitutes another counterexample containing more zero entries than C does).

Since C and C^T are monotone matrices and since each row and each column contains at least one zero entry, there exist indices i and j such that $c_{ij} > 0$ and (i) $c_{i-1,j} = 0$ and $c_{i,j+1} = 0$ or (ii) $c_{i+1,j} = 0$ and $c_{i,j-1} = 0$ holds. W.l.o.g. suppose that $c_{i-1,j} = 0$ and $c_{i,j+1} = 0$ (the other case is symmetric). We show that $c_{ij} \leq c_{pq}$ for all $i \leq p \leq m$ and $1 \leq q \leq j$. Indeed, since $c_{i-1,q} + c_{ij} \leq c_{i-1,j} + c_{iq}$ and $c_{i-1,j} = 0$, $c_{iq} \geq c_{ij}$ holds for all $q < j$. An analogous argument yields $c_{pj} \geq c_{ij}$ for all $p > i$. Finally for $p > i$ and $q < j$, $c_{ij} + c_{pq} \geq c_{iq} + c_{pj} \geq 2c_{ij}$ holds. Summarizing, this yields $c_{pq} \geq c_{ij}$ for all $p \geq i$ and $q \leq j$.

Consider the matrix $C' = C - c_{ij} L^{(ij)}$. C' is nonnegative and (since $c'_{ij} = 0$) it contains more zero entries than C. We claim that C' is again a Monge matrix, thus derive a contradiction to the choice of C and prove the lemma. Suppose that C' is not a Monge matrix. Then there exist indices $1 \leq p \leq m$ and $1 \leq q \leq n$ with $c'_{pq} + c'_{p+1,q+1} > c'_{p,q+1} + c'_{p+1,q}$. The only interesting case arises if $p = i-1$ and $q = j$. By the construction, then $c'_{pq} = c'_{p+1,q+1} = c'_{p,q+1} = 0$ and $c'_{p+1,q} \geq 0$ holds. □

The above lemma shows that each nonnegative Monge matrix can be written as a nonnegative linear combination of matrices in $\mathscr{M}$, i.e. that $\mathscr{M}$ is a superset of the extremal rays of the cone of nonnegative Monge matrices. The following lemma proves that $\mathscr{M}$ indeed equals the set of extremal rays of this cone.

Lemma 2.5: The nonnegative scalar multiples of the matrices in $\mathscr{M}$ form extremal rays of the cone of nonnegative Monge matrices.

Proof: It is sufficient to show that no matrix $C \in \mathscr{M}$ can be represented as nonnegative linear combination of the other matrices in $\mathscr{M}$. Clearly, neither any $H^{(i)}$ nor any $V^{(j)}$ may be written as an nonnegative linear combination of the other matrices in $\mathscr{M}$.

Next consider some $L^{(ij)}$ and suppose that there exists a non trivial nonnegative linear combination of $L^{(ij)}$ with matrices in $\mathscr{M}$. Since the first row and the last column of $L^{(ij)}$ only contains zero entries, no matrix in $\mathscr{H} \cup \mathscr{V} \cup \mathscr{R}$ may contribute to the nonnegative linear combination for $L^{(ij)}$. Since $l_{ij}^{(ij)} = 1$, at least one matrix $L^{(rs)}$ with $r < i$ and $j \geq s$ or $r \leq i$ and $j > s$ must have a nonzero coefficient in the nonnegative linear combination of $L^{(ij)}$. Since $l_{rs}^{(ij)} = 0$, this yields a contradiction. Matrices in $\mathscr{R}$ are handled symmetrically. □

Next, the nonnegativity constraint is removed and Monge matrices with arbitrary real entries are considered.

Theorem 2.6: (Characterization of Monge matrices). For every $m \times n$ Monge matrix C there exist real numbers λ_j and nonnegative numbers κ_i and v_{ij} such that

$$C = \sum_{i=1}^{m} \kappa_i H^{(i)} + \sum_{j=1}^{n} \lambda_j V^{(j)} + \sum_{i=1}^{m-1} \sum_{j=2}^{n} v_{ij} R^{(ij)} .$$

Proof: Each Monge matrix C can be transformed into a nonnegative Monge matrix by adding a very large constant α to every entry, cf. Observation 2.1.(iv). This procedure corresponds to adding $\alpha \sum_{i=1}^{n} V^{(i)}$ to matrix C. Moreover,

$$L^{(ij)} = R^{(i-1,j+1)} + \sum_{p=i}^{m} H^{(p)} - \sum_{q=j+1}^{n} V^{(q)}$$

holds. Combining this with the statement in Lemma 2.4 completes the proof. □

Finally, *symmetric* Monge matrices are investigated. Define for all $1 \le i \le n$

$$S^{(i)} = H^{(i)} + V^{(i)}$$

and for all $2 \le i \le n$ and $1 \le j \le n-1$

$$T^{(ij)} = L^{(ij)} + R^{(ji)} .$$

All matrices $S^{(i)}$ and $T^{(ij)}$ are symmetric Monge matrices. Moreover for $i \le j$, $T^{(ij)} = T^{(j+1,i-1)} + \sum_{p=i}^{j} S^{(p)}$ holds. These observations and an argument analogous to the proof of Lemma 2.4 yield the following characterization.

Theorem 2.7: (Characterization of symmetric Monge matrices). For every symmetric $n \times n$ Monge matrix C, there exist real numbers κ_i and nonnegative numbers ν_{ij} such that

$$C = \sum_{i=1}^{n} \kappa_i S^{(i)} + \sum_{i=2}^{n} \sum_{j=1}^{i-1} \nu_{ij} T^{(ij)} . \tag{2}$$

□

3 Applications

This section deals with optimization problems (P) of the following type: For a set $\mathscr{F}$ of $m \times n$ matrices and an $m \times n$ cost-matrix C, find that matrix $F = (f_{ij})$ in $\mathscr{F}$ that minimizes the Hadamard product of F with C, i.e. the sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} f_{ij} c_{ij} .$$

In general, the set $\mathscr{F}$ will be defined implicitly and not by enumeration of its elements. The arguments in this section are based on the following trivial observation (an analogous statement holds for symmetric Monge matrices).

Observation 3.1: Let (P) *be an optimization problem defined as above and assume that the same matrix $F_0 \in \mathcal{F}$ yields the optimal solution to problem* (P) *for any cost-matrix in $\mathcal{M}$. Then for any nonnegative Monge matrix C, matrix F_0 also constitutes an optimal solution to* (P) *with cost-matrix C.* □

The *traveling salesman problem* TSP is defined as follows: Given n cites and an $n \times n$ distance matrix C, find a *tour* of minimal cost, i.e. find the minimum $\min_\phi \{c(\phi): \phi$ is a cyclic permutation$\}$, where $c(\phi)$ denotes the overall cost of a tour and is defined by $c(\phi) = \sum_{i=1}^{n} c_{i\phi(i)}$. Since the TSP is known to be NP-hard, scientific research is interested in polynomial time solvable special cases of the TSP.

Supnick [7] has shown that for a symmetric $n \times n$ Monge matrix, there always exists a tour of minimal cost that has the form $\langle 1, 3, 5, \ldots, 6, 4, 2 \rangle$ ("visit the odd cities in increasing order and afterwards the even cities in decreasing order") and that there always exists a tour of maximum cost of the form $\langle n, 2, n-2, 4, n-4, 6, \ldots, 5, n-3, 3, n-1, 1 \rangle$. The latter result was obtained independently by Michalski [4]. In case $\mathcal{F}$ consists of all cyclic permutation matrices, the optimization problem (P) turns into the traveling salesman problem. In the following, Observation 3.1 is applied to the TSP and simple alternative proofs for these two results of Supnick are derived.

Theorem 3.2: (*Supnick* [7]) *Let C be an $n \times n$ symmetric Monge matrix. Then for the TSP with distance matrix C,*

(i) $\langle 1, 3, 5, \ldots, 6, 4, 2 \rangle$ *is the minimum cost tour, and*
(ii) $\langle n, 2, n-2, 4, n-4, 6, \ldots, 5, n-3, 3, n-1, 1 \rangle$ *is the maximum cost tour.*

Proof: We only prove statement (i); the proof of statement (ii) can be done analogously. Due to Observation 3.1 and Theorem 2.7, it is sufficient to prove that the *Supnick permutation* $\phi = \langle 1, 3, 5, \ldots, 6, 4, 2 \rangle$ is optimal for all matrices $S^{(i)}$, $1 \le i \le n$, and for all matrices $T^{(ij)}$, $2 \le i \le n$ and $1 \le j < i$. Matrices $S^{(i)}$ are *sum matrices*, and hence for them the length of a tour does not depend on the tour's combinatorial structure and is constant. Therefore, every cyclic permutation constitutes an optimum tour for $S^{(i)}$.

Next, consider some matrix $T^{(ij)}$, $2 \le i \le n$ and $1 \le j < i$. Partition the indices of $T^{(ij)}$ into three sets $I = \{1, \ldots, j\}$, $J = \{j+1, \ldots, i-1\}$ and $K = \{i, \ldots, n\}$. I and K are both non-empty sets, whereas J might be empty. Then the costs of all edges between cities in I and cities in K are one, and all other edges cost nothing. We distinguish three cases depending on the size of $|J|$.

(Case 1). If $|J| \ge 2$ then $c(\phi) = 0$ and $c(\pi) \ge 0$ for all tours π.

(Case 2). If $|J| = 1$ then $c(\phi) = 1$. Every tour must contain at least one edge connecting I to K (the tour must go from I to K and return to I; in the best case, one of these transitions is via J and the other has cost one). Hence, $c(\pi) \geq 1$ for all tours π.

(Case 3). If $|J| = 0$, then $c(\phi) = 2$. Since every tour π must use at least two edges connecting I and K, $c(\pi) \geq 2$ holds.

Summarizing, ϕ is in any case an optimal tour. $\square$

The linear assignment problem consists in finding a (not necessarily cyclic) permutation π for an $n \times n$ cost-matrix $C = (c_{ij})$ that minimizes the sum $\sum_{i=1}^{n} c_{i\pi(i)}$. It is a well-known fact (see e.g. [1]) that the identity permutation yields an optimum solution for the linear assignment problem on Monge matrices. Combining Observation 3.1 with similar (but even simpler) arguments as in the proof of Theorem 3.2 yields a short proof for this fact.

4 Conclusion

In this note we investigated the cone of Monge matrices and thereby derived a simple additive characterization of Monge matrices. We showed how to apply this characterization to get a simple proof for an ancient result of Supnick (1957) on traveling salesman tours for symmetric Monge matrices.

Concerning further research, we note that there are many results in the Soviet literature on the TSP with "specially structured" distance matrices. Some of these matrix classes are cones. It might be interesting to determine the extremal rays of these cones and to derive simple proofs for these cases by exploiting the structure of the cone.

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