Available online at www.sciencedirect.com
ScienceDirect
Operations Research Letters 36 (2008) 43–45
Operations Research Letters
www.elsevier.com/locate/orl

Minimum L_k path partitioning—An illustration of the Monge property

Pei-Hao Ho[a], Arie Tamir[b], Bang Ye Wu[a,*]

[a]Department of Computer Science and Information Engineering, Shu-Te University, YenChau, KaoShiung 824, Taiwan, ROC
[b]Department of Statistics and Operations Research, School of Mathematical Sciences, Tel Aviv University, Israel

Received 18 March 2007; accepted 19 March 2007
Available online 4 July 2007

Abstract

We investigate the problem of cutting a given sequence of positive real numbers into p pieces, with the objective of minimizing the L_k norm of the sums of the numbers in these pieces. We observe that this problem satisfies a Monge property, and thus derive fast algorithms for it.

Keywords: Partition; Dynamic programming; Convexity; Monge property

1. Introduction

Let $S=(x_1,\ldots,x_n)$ be a sequence of n positive real numbers. Given a positive integer p and a real $k\geqslant 1$, we consider the problem of partitioning S into p nonempty subsequences $\{S_1,S_2,\ldots,S_p\}$ of S, of *consecutively indexed* numbers, such that the L_k norm is minimized. Formally, a p-partition $\Pi=\{S_1,S_2,\ldots,S_p\}$ of S is defined by a set of indices $\{h(1),h(2),\ldots,h(p),h(p+1)\}$, satisfying $1=h(1)<h(2)<\cdots<h(p)<h(p+1)=n+1$, where $S_j=(x_{h(j)},x_{h(j)+1},\ldots,x_{h(j+1)-1})$, for $j=1,\ldots,p$. The L_k norm of Π is defined by $\sum_{1\leqslant j\leqslant p}|\sigma(S_j)-\mu|^k$, where $\sigma(S_j)=x_{h(j)}+x_{h(j)+1}+\cdots+x_{h(j+1)-1}$, and $\mu=(\sum_{1\leqslant j\leqslant p}\sigma(S_j))/p=(\sum_{1\leqslant i\leqslant n}x_i)/p$. The L_∞ norm of Π is defined by $\max_{1\leqslant j\leqslant p}|\sigma(S_j)-\mu|$.

The problem is known as the *minimum L_k path partitioning problem* [9,10,13,15], in which the input numbers are thought of as the weights of vertices of a path. This path equipartitioning problem has several applications. For example, in image processing [10] we may want to transform a picture of n gray-levels into p gray-levels ($p<n$). The transformation can be done by cutting an n-vertex path into p subpaths. If the ith vertex of the path is in the jth subpath, all pixels of gray-level i will be assigned a new gray-level j. One hopes that, in the resulting picture, the number of pixels of different gray-levels are as equal as possible. The L_k norm of a partition is a measurement of how equal a partition is. For example, the L_2 norm is actually the *variance* of the numbers of pixels of all gray-levels.

The minimum L_k path partitioning problem for any finite constant k, as well as $k=\infty$, can be modeled by a recurrence equation, and solved in $O(pn^2)$ time by using dynamic programming. For some values of k more efficient algorithms are known. Specifically, an $O(pn)$-time algorithm for the L_1 norm case is presented in [13,15], and an $O(pn\log p)$ time algorithm for the L_∞ norm is reported in [9]. We also cite [10], where a related equipartition problem, called the *most uniform path partitioning* problem was introduced and solved in $O(pn^2)$-time. In this model the objective is to minimize the difference between the maximum and the minimum weights of the subpaths.

In this note we show that for any real number $k\geqslant 1$, the minimum L_k path partitioning problem can be solved in $O(pn)$ time by applying the algorithms developed in [1,4,5,7,16]. Specifically, we illustrate that the above partitioning problem satisfies the well known convex Monge property.

In the last section we briefly discuss the complexity of solving some related partitioning problems, including some instances of the minimum L_k path partitioning problem, with $0<k<1$.

2. The Monge property of the L_k path partitioning problem, $k\geqslant 1$

In the following, we relax the above definition of μ as $\mu=(\sum_{1\leqslant i\leqslant n}x_i)/p$, and instead assume that μ is any real

* Corresponding author. Tel.: +886 761580004612; fax: +886 761580004699.
E-mail address: bangye@mail.stu.edu.tw (B.Y. Wu).

0167-6377/$ - see front matter © 2007 Elsevier B.V. All rights reserved.
doi:10.1016/j.orl.2007.03.010

number, which is independent of the p-partition that we select. In particular, μ can be any constant which is independent of the elements in S, e.g., $\mu = 0$.

For $i = 1, \ldots, n$, and $j = 1, \ldots, i$, define $c(i, j)$ to be the minimum L_k cost of partitioning the prefix subsequence $(x_1, x_2, \ldots, x_i)$ into j consecutive parts. Set $c(i, j) = \infty$ if $j > i$. For $r < i$, let $X_{ri} = x_{r+1} + x_{r+2} + \cdots + x_i$. By definition $c(i, 1) = |X_{0i} - \mu|^k$ for each $i \leqslant n$, and for $1 < j \leqslant i$, we clearly have

$$c(i, j) = \min_{r<i}\{c(r, j-1) + |X_{ri} - \mu|^k\}. \tag{1}$$

Using dynamic programming, we can obtain $c(n, p)$, the optimal solution value to the model, by iteratively computing $c(i, j)$ for $j = 2, \ldots, p$, and $i = j, \ldots, n$. The time complexity is $O(pn^2)$. Note that X_{ri} for any $0 \leqslant r < i \leqslant n$, can be obtained in constant time after a linear time preprocessing stage for computing the prefix sum X_{0i} for each $i \leqslant n$.

The above algorithm iteratively computes $c(i, j)$ with j varying from small to large. Therefore, we can assume that all the values $c(i, j-1)$, $i = 1, \ldots, n$, are already available when considering a specific j. We will next show that for each $1 < j \leqslant p$, all the terms $c(i, j)$, $1 \leqslant i \leqslant n$, can be found in $O(n)$ time. That will result in an $O(pn)$ algorithm for the minimum L_k path partition problem. The improvement is implied by observing that the function $|X_{ri} - \mu|^k$ satisfies the following quadrangle (supermodular) inequality, also known as the convex Monge property. (We follow the definitions in [5]. Note that the definitions of convexity and concavity in terms of the Monge property have been interchanged in some references.)

Definition 1. A real function $w(a, b)$ defined on the integers $1 \leqslant a \leqslant b \leqslant n$ has the convex Monge property if $w(a, p) + w(b, q) \leqslant w(b, p) + w(a, q)$ for all $a \leqslant b \leqslant p \leqslant q$. It has the concave Monge property if $w(a, p) + w(b, q) \geqslant w(b, p) + w(a, q)$ for all $a \leqslant b \leqslant p \leqslant q$.

Lemma 1. *Let f be a real convex function defined on the set R of real numbers. Then the function $w(r, i) = f(X_{ri})$ satisfies the convex Monge property.*

Proof. For all $0 \leqslant a \leqslant b \leqslant p \leqslant q \leqslant n$, the quadrangle (convex) inequality $w(a, p) + w(b, q) \leqslant w(b, p) + w(a, q)$ is equivalent to $w(b, q) - w(b, p) \leqslant w(a, q) - w(a, p)$, which, by definition, is equivalent to

$$f(X_{bq}) - f(X_{bp}) \leqslant f(X_{aq}) - f(X_{ap}).$$

The latter inequality follows from the convexity of the function f, and the facts that $X_{bq} - X_{bp} = X_{aq} - X_{ap}$, and $X_{bq} \leqslant X_{aq}$. $\square$

To prove our claim that the function $w(r, i) = |X_{ri} - \mu|^k$ satisfies the convex Monge property, we observe that the real function $f(y) = |y - \mu|^k$ is convex for any constants μ and $k \geqslant 1$.

Finally, since the cost function in (1) satisfies the convex Monge property, the algorithms in [1,4,5,7,16] can be applied to the problem.

Theorem 2. *For any real constants μ and $k \geqslant 1$, the minimum L_k path partition problem can be solved in $O(pn)$ time.*

3. Related models

We briefly comment on the solvability of related path partitioning problems.

First consider the L_k path partitioning problem when $0 < k < 1$. In this case the real function $f(y) = |y - \mu|^k$ is concave over the interval $[\mu, \infty)$, as well as over the interval $(-\infty, \mu]$. However, it is not globally concave. Nevertheless, if $\mu = 0$, the concavity of $f(y) = y^k$ over the nonnegative real ray implies that in this case, the function $w(r, i) = X_{ri}^k$ satisfies the concave Monge property. Hence, we can use the algorithm in [8] to solve the respective model.

Theorem 3. *For any real constants, $0 < k < 1$, and $\mu \leqslant \min_{1 \leqslant i \leqslant n} x_i$, the minimum L_k path partition problem can be solved in $O(pn\,\alpha(n))$ time. ($\alpha(n)$ is the functional inverse of the Ackermann's function.)*

Next consider the L_∞ path partitioning model. As mentioned above, an $O(pn \log p)$-time algorithm for this model (with $\mu = (\sum_{1 \leqslant i \leqslant n} x_i)/p$), is reported in [9]. We note that if $\mu = 0$, the model is actually solvable in optimal $O(n)$ effort by the algorithms in [2,3]. In fact, if all the prefix sums X_{0i}, $i = 1, \ldots, n$, are already known, this minmax problem can be solved in $O(p^2 \log^2 n)$ time by adapting the algorithm in [12]. The latter bound is sublinear in n for relatively small values of p, i.e., $p = o(n^{1/2}/\log n)$. In the context of general minmax (L_∞ norm) path p-partitioning models, the reader is referred to [11,14].

For applications of the Monge property to other path partitioning problems see [6].

Acknowledgments

Bang Ye Wu and Pei-Hao Ho were supported in part by an NSC Grant 94-2213-E-366-006 from the National Science Council, Taiwan. The authors thank Gerhard Woeginger and Jan Karel Lenstra for their contribution to this collaboration.

References

[1] D. Eppstein, Sequence comparison with mixed convex and concave costs, J. Algorithms 11 (1990) 85–101.

[2] G.N. Frederickson, Optimal algorithms for partitioning trees and locating p-centers in trees, Technical Report CSD-TR-1029, Purdue University, 1990.

[3] G.N. Frederickson, Optimal algorithms for tree partitioning, in: Proceedings of the Second ACM-SIAM Symposium on Discrete Algorithms, San Francisco, 1991, pp. 168–177.

[4] Z. Galil, K. Park, A linear time algorithm for concave one-dimensional dynamic programming, Inform. Process. Lett. 33 (1989/1990) 309–311.

[5] Z. Galil, K. Park, Dynamic programming with convexity, concavity and sparsity, Theor. Comput. Sci. 92 (1) (1992) 49–76.

[6] R. Hassin, A. Tamir, Improved complexity bounds for location problems on the real line, Oper. Res. Lett. 10 (1991) 395–402.

[7] M.M. Klawe, A simple linear time algorithm for concave one-dimensional dynamic programming, Technical Report, UMI Order Number: TR-89-16, University of British Columbia, 1989.
[8] M.M. Klawe, D.J. Kleitman, An almost linear time algorithm for generalized matrix searching, SIAM J. Discrete Math. 3 (1990) 81–97.
[9] M. Liverani, A. Morgana, B. Simeone, G. Storchi, Path equipartition in the Chebyshev norm, Eur. J. Oper. Res. 123 (2000) 428–436.
[10] M. Lucertini, Y. Perl, B. Simeone, Most uniform path partitioning and its use in image processing, Discrete Appl. Math. 42 (1993) 227–256.
[11] F. Manne, T. Sorevik, Optimal partitioning of sequences, J. Algorithms 19 (1995) 235–249.
[12] N. Megiddo, A. Tamir, An $O(p^2 \log^2 n)$ algorithm for the unweighted p-center problem on the line, unpublished Technical Report, 1981 (revised 1991).
[13] G. Messe, Partitions of chains with minimum imbalance, Cah. Cent. Etud. Rech. Operationelle 27 (1985) 235–245.
[14] B. Olstad, F. Manne, Efficient partitioning of sequences, IEEE Trans. Comput. 44 (1995) 1322–1326.
[15] C.D. Simone, M. Lucertini, S. Pallotino, B. Simeone, Fair dissections of spiders, worms and caterpillars, Networks 20 (1990) 323–344.
[16] R. Wilber, The concave least-weight subsequence problem revisited, J. Algorithms 9 (1988) 418–425.