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# ON TESTING CONVEXITY AND SUBMODULARITY*

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**Abstract.** Convex and submodular functions play an important role in many applications, and in particular in combinatorial optimization. Here we study two special cases: convexity in one dimension and submodularity in two dimensions. The latter type of functions are equivalent to the well-known *Monge matrices*. A matrix $V = \{v_{i,j}\}_{i,j=0}^{i=n_1,j=n_2}$ is called a Monge matrix if for every $0 \le i < i' \le n_1$ and $0 \le j < j' \le n_2$ we have $v_{i,j} + v_{i',j'} \le v_{i,j'} + v_{i',j}$. If inequality holds in the opposite direction, then $V$ is an *inverse Monge* matrix (supermodular function). Many problems, such as the traveling salesperson problem and various transportation problems, can be solved more efficiently if the input is a Monge matrix.

In this work we present testing algorithms for the above properties. A testing algorithm for a predetermined property $\mathcal{P}$ is given query access to an unknown function $f$ and a distance parameter $\epsilon$. The algorithm should accept $f$ with high probability if it has the property $\mathcal{P}$ and reject it with high probability if more than an $\epsilon$-fraction of the function values should be modified so that $f$ obtains the property. Our algorithm for testing whether a 1-dimensional function $f : [n] \to \mathbb{R}$ is convex (concave) has query complexity and running time of $O((\log n)/\epsilon)$. Our algorithm for testing whether an $n_1 \times n_2$ matrix $V$ is a Monge (inverse Monge) matrix has query complexity and running time of $O((\log n_1 \cdot \log n_2)/\epsilon)$.

**Key words.** property testing, convex functions, Monge matrices, approximation algorithms, randomized algorithms

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## 1. Introduction.

Convex functions and their combinatorial analogues, submodular functions, play an important role in many disciplines and applications, including combinatorial optimization, game theory, probability theory, and electronic trade. Such functions exhibit a rich mathematical structure (see Lovász [Lov83]), which often makes it possible to efficiently find their minimum [GLS81, IFF01, Sch00] and thus leads to efficient algorithms for many important optimization problems. Convex functions over discrete domains are defined as follows.

DEFINITION 1 (convex and concave). *Let $f$ be a function defined over a discrete domain $X$. The function $f$ is* convex *if for all $x, y \in X$ and for all $0 \le \alpha \le 1$ such that $\alpha x + (1-\alpha)y \in X$, it holds that $f(\alpha x + (1-\alpha)y) \le \alpha f(x) + (1-\alpha)f(y)$. The function $f$ is* concave *if for all $x, y \in X$ and for all $0 \le \alpha \le 1$ such that $\alpha x + (1-\alpha)y \in X$, it holds that $f(\alpha x + (1-\alpha)y) \ge \alpha f(x) + (1-\alpha)f(y)$.*

Submodular functions are defined as follows: Let $\mathcal{I} = I_1 \times I_2 \times \cdots \times I_d$, $d \ge 2$, be a product space where $I_q \subseteq \mathbb{R}$. In particular, we are interested in discrete domains $I_q = \{0, \ldots, n_q\}$. The *join* and *meet* operations are defined for every $x, y \in \mathcal{I}$ as follows:

$$(x_1, \ldots, x_d) \vee (y_1, \ldots, y_d) \stackrel{\text{def}}{=} (\max\{x_1, y_1\}, \ldots, \max\{x_d, y_d\})$$

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and

$$(x_1, \ldots, x_d) \wedge (y_1, \ldots, y_d) \stackrel{\text{def}}{=} (\min\{x_1, y_1\}, \ldots, \min\{x_d, y_d\}),$$

respectively.

DEFINITION 2 (submodularity and supermodularity). *A function $f : \mathcal{I} \to \mathbb{R}$ is* submodular *if for every $x, y \in \mathcal{I}$, $f(x \vee y) + f(x \wedge y) \leq f(x) + f(y)$. The function $f$ is* supermodular *if for every $x, y \in \mathcal{I}$, $f(x \vee y) + f(x \wedge y) \geq f(x) + f(y)$.*

Certain subclasses of submodular functions are of particular interest. One such subclass is that of *submodular set* functions, which are defined over binary domains. That is, $I_q = \{0, 1\}$ for every $1 \leq q \leq d$, and so each $x \in \mathcal{I}$ corresponds to a subset of $\{1, \ldots, d\}$. Such functions are used, for example, in the scenario of combinatorial auctions on the Internet (e.g., [dVV00, LLN01]).

Another important subclass is the class of *Monge* functions, which are obtained when the domain is large but the dimension is $d = 2$. Since such functions are 2-dimensional, it is convenient to represent them as 2-dimensional matrices, which are referred to as *Monge matrices*. When the function is a 2-dimensional supermodular function the corresponding matrix is called an *inverse Monge matrix*.

The first problem that was shown to be solvable more efficiently if the underlying cost matrix is a Monge matrix is the classical Hitchcock transportation problem (see Hoffman [Hof63]). Since then it has been shown that many other combinatorial optimization problems can be solved more efficiently in this case (e.g., weighted bipartite matching and NP-hard problems such as the traveling salesperson problem). See [BKR96] for a comprehensive survey on Monge matrices and their applications.

**1.1. Testing convexity and submodularity.** In this paper we approach the questions of convexity and submodularity from within the framework of property testing [RS96, GGR98]. (For surveys on property testing see [Ron01, Fis01].) Let $f$ be a fixed but unknown function, and let $\mathcal{P}$ be a fixed property of functions (such as the convexity or submodularity of a function). A testing algorithm for the property $\mathcal{P}$ should determine, by querying $f$, whether $f$ has the property $\mathcal{P}$ or whether it is $\epsilon$-*far* from having the property for a given distance parameter $\epsilon$. By $\epsilon$-*far* we mean that more than an $\epsilon$-fraction of the values of $f$ should be modified so that $f$ obtains the desired property $\mathcal{P}$.

*Our results.* We present efficient testing algorithms for discrete convexity in one dimension and for Monge matrices. Specifically, we do the following:

- We describe and analyze an algorithm that tests whether a function $f : [n] \to \mathbb{R}$ is convex (concave). The running time of this algorithm is $O(\log n/\epsilon)$.
- We describe and analyze a testing algorithm for Monge and inverse Monge matrices whose running time is $O((\log n_1 \cdot \log n_2)/\epsilon)$ when given an $n_1 \times n_2$ matrix.

  Furthermore, the testing algorithm for inverse Monge matrices can be used to derive a testing algorithm, with the same complexity, for an important subfamily of Monge matrices called *distribution matrices*. A matrix $V = \{v_{i,j}\}$ is said to be a distribution matrix if there exists a nonnegative *density matrix* $D = \{d_{i,j}\}$ such that every entry $v_{i,j}$ in $V$ is of the form $v_{i,j} = \sum_{k \leq i} \sum_{\ell \leq j} d_{k,\ell}$. In other words, the entry $v_{i,j}$ corresponds to the cumulative density of all entries $d_{k,\ell}$ such that $k \leq i$ and $\ell \leq j$.

In both cases the complexity of the algorithms is linear in $1/\epsilon$ and polylogarithmic in the size of the domain.

### 1.2. Techniques.

*Convexity in one dimension.* We start with the following basic observation: A function $f : [n] \to \mathbb{R}$ is convex if and only if for every $1 \leq i \leq n-1$, $(f(i+1) - f(i)) - (f(i) - f(i-1)) \geq 0$. Given this characterization, consider the *difference* function $f'$, which is defined as $f'(i) = f(i) - f(i-1)$. The function $f'$ can be viewed as the discrete analogue of the first derivative of $f$. By the above observation we have that $f$ is convex if an only if $f'$ is monotone nondecreasing. Hence, a tempting approach for testing whether $f$ is convex would be to test whether $f'$ is monotone nondecreasing, where this can be done in time $O(\log n/\epsilon)$ [EKK+00, BRW99, DGL+99].

Unfortunately, this approach does not work. There are functions $f$ that are very far from convex, but their difference function $f'$ is very close to monotone.[1] Therefore, instead of considering only consecutive points $i, i+1$, we consider pairs of points $i, j \in [n]$ that are not necessarily consecutive. More precisely, we select intervals $\{i, \ldots, j\}$ of varying lengths and check that for each interval selected, certain constraints are satisfied. If $f$ is convex, then these constraints are satisfied for every interval. On the other hand, we show that if $f$ is $\epsilon$-far from convex, then the probability that we observe a violation of some constraint is sufficiently large.

*Monge matrices.* As stated above, it is convenient to represent 2-dimensional submodular functions as 2-dimensional Monge matrices. Thus a function $f : \{0, \ldots, n_1\} \times \{0, \ldots, n_2\} \to \mathbb{R}$ can be represented as the matrix $V = \{v_{i,j}\}_{i,j=0}^{i=n_1,j=n_2}$, where $v_{i,j} = f(i,j)$. Observe that for every pair of indices $(i,j'), (i',j)$ such that $i < i'$ and $j < j'$ we have that $(i,j') \vee (i',j) = (i',j')$ and $(i,j') \wedge (i',j) = (i,j)$. It follows from Definition 2 that $V$ is a Monge matrix ($f$ is a 2-dimensional submodular function) if and only if

$$\forall i, j, i', j' \quad \text{s.t.} \quad i < i',\ j < j' \ : \qquad v_{i,j} + v_{i',j'} \leq v_{i,j'} + v_{i',j}$$

and $V$ is an inverse Monge matrix ($f$ is a 2-dimensional supermodular function) if and only if

$$\forall i, j, i', j' \quad \text{s.t.} \quad i < i',\ j < j' \ : \qquad v_{i,j} + v_{i',j'} \geq v_{i,j'} + v_{i',j}\ .$$

That is, in both cases we have a constraint for every quadruple $v_{i,j}$, $v_{i',j'}$, $v_{i,j'}$, $v_{i',j}$ such that $i < i'$ and $j < j'$.[2] Our algorithm selects such quadruples according to a particular (nonuniform) distribution and verifies that the constraint is satisfied for every quadruple selected. Clearly, the algorithm always accepts Monge matrices. The main thrust of the analysis is in showing that if the matrix $V$ is far from being Monge, then the probability of obtaining a "bad" quadruple is sufficiently large.

A central building block in proving the above is the following combinatorial problem, which may be of independent interest. Let $C$ be a given matrix, possibly containing negative values, and let $R$ be a subset of positions in $C$. We are interested in refilling the entries of $C$ that reside in $R$ with *nonnegative* values such that the following constraint is satisfied: for every position $(i,j)$ that does not belong to $R$, the sum of the modified values in $C$ that are below[3] $(i,j)$ is the same as in the original

[1] In particular, consider the function $f$ such that for every $i \leq n/2$, $f(i) = i$, and for $i > n/2$, $f(i) = i - 1$. In other words, $f'(i) = 1$ for every $i$ except $i = n/2$, where $f'(i) = 0$. Then $f'$ is very close to monotone, but it is not hard to verify that $f$ is far from convex.

[2] It is easy to verify that for all other $i, j, i', j'$ (with the exception of the symmetric case where $i' < i$ and $j' < j$), the constraint holds trivially (with equality).

[3] We denote the lower left position of the matrix $C$ by $(0,0)$.

matrix $C$. That is, the sum of the modified values in entries $(k, \ell)$ such that $k \leq i$ and $j \leq \ell$ remains as it was.

We provide sufficient conditions on $C$ and $R$ under which the above is possible and describe the corresponding procedure that refills the entries of $C$ that reside in $R$. Our starting point is a simple special case in which $R$ corresponds to a submatrix of $C$. In such a case it suffices that for each row and each column in $R$, the sum of the corresponding entries in the original matrix $C$ is nonnegative. Under these conditions a simple greedy algorithm can modify $C$ as required. Our procedure for general subsets $R$ is more involved but uses the submatrix case as a subroutine.

**1.3. Further research.** We suggest the following open problems. First, it remains open to determine the complexity of testing discrete convexity (concavity) when the dimension $d$ of the input domain is greater than 1 and for testing submodular (supermodular) functions when the dimension $d$ is greater than 2. Note that though submodular functions can be viewed as a certain interpretation of convexity in dimensions $d \geq 2$, they do not necessarily satisfy Definition 1.

It seems that our algorithm for testing Monge matrices and its analysis can be extended to work for testing the special case of distribution matrices of dimension $d > 2$, where the complexity of the resulting algorithm is $O((\prod_{q=1}^{d} \log n_q)/\epsilon)$. However, as opposed to the $d = 2$ case, where Monge matrices are only slightly more general than distribution matrices, for $d > 2$ Monge matrices are more expressive. Hence it is not immediately clear how to adapt our algorithm to testing Monge matrices in higher dimensions.

It would also be interesting to find an efficient testing algorithm for the subclass of submodular set functions, which are defined over binary domains.

Finally, in many optimization problems it is enough that the underlying cost matrix is a permutation of a Monge matrix. In such cases it may be useful to test whether a given matrix is a permutation of some Monge matrix or far from any permuted Monge matrix.

*Organization.* The testing algorithm for convexity is described in section 2. The remainder of the paper is dedicated to testing Monge matrices. In section 3 we describe several building blocks that will be used by our testing algorithm for Monge matrices. In section 4 we describe a testing algorithm for Monge matrices whose complexity is $O(n/\epsilon)$, where we assume for simplicity that the matrix is $n \times n$. Building on this algorithm and its analysis, in section 5 we present a significantly faster algorithm whose complexity is $O\left((\log^2 n)/\epsilon\right)$. We conclude this section with a short discussion concerning distribution matrices.

**2. Testing convexity in one dimension.** As noted in the introduction, in the case that the domain is $X = [n] = \{0, \ldots, n\}$, we get the following characterization for convexity, whose proof is included for completeness.

CLAIM 1. *A function $f : [n] \to \mathbb{R}$ is* convex *if and only if for all $1 \leq i \leq n-1$, $f(i) - f(i-1) \leq f(i+1) - f(i)$.*

*Proof.* If $f$ is convex, then in particular for $x = i - 1$, $y = i + 1$, and $\alpha = 1/2$ we have $\alpha x + (1-\alpha)y = \frac{i-1}{2} + \frac{i+1}{2} = i$. By Definition 1, $f(i) \leq \frac{1}{2}f(i-1) + \frac{1}{2}f(i+1)$, or, equivalently, $f(i) - f(i-1) \leq f(i+1) - f(i)$.

In the other direction, suppose that $f(i) - f(i-1) \leq f(i+1) - f(i)$ for every $1 \leq i \leq n-1$. Consider any $x, y \in [n]$ and $0 < \alpha < 1$ such that $z = \alpha \cdot x + (1-\alpha) \cdot y$ is an integer. Assume without loss of generality that $x < y$. Now we have that

$$f(y)-f(y-1) \geq f(y-1)-f(y-2) \geq \cdots \geq f(z+1)-f(z) \geq f(z)-f(z-1) \geq \cdots \geq f(x+1)-f(x).$$

Then, since the differences are monotone nonincreasing, the average of the first $\alpha(y-x)$ differences is greater than or equal to the average of the next $(1-\alpha)(y-x)$ differences. Since $z = y - \alpha(y-x) = x + (1-\alpha)(y-x)$, we have that

$$\frac{(f(y)-f(y-1))+(f(y-1)-f(y-2))+\cdots+(f(z+1)-f(z))}{\alpha(y-x)} \tag{1}$$

$$\geq \frac{(f(z)-f(z-1))+(f(z-1)-f(z-2))+\cdots+(f(x+1)-f(x))}{(1-\alpha)(y-x)}. \tag{2}$$

This is equivalent to $(1-\alpha)(f(y)-f(z)) \geq \alpha(f(z)-f(x))$; that is, $f(z) \leq \alpha f(x) + (1-\alpha)f(y)$ as required. □

Denote by $I_{i,j}$ the interval $\{i, i+1, \ldots, j\}$ of points. Let $mid = \lfloor (i+j)/2 \rfloor$ be the midpoint of $I_{i,j}$.

DEFINITION 3. *For every $0 \leq i < j \leq n$ such that $j - i > 7$, we say that the interval $I_{i,j}$ is* good with respect to $f$ *if the following holds:*

$$f(i+1)-f(i) \leq \frac{f(mid-1)-f(i+1)}{(mid-1)-(i+1)} \leq f(mid)-f(mid-1) \leq f(mid+1)-f(mid)$$
$$\leq f(mid+2)-f(mid+1) \leq \frac{f(j-1)-f(mid+2)}{(j-1)-(mid+2)} \leq f(j)-f(j-1).$$

*Otherwise, we say that the interval is* bad with respect to $f$. *If $j - i \leq 7$, then $I_{i,j}$ is good with respect to $f$ if and only if the function $f$ is convex over $I_{i,j}$.*

In order to test if $f$ is convex we test recursively if subintervals of $I_{0,n}$ are good.

ALGORITHM 1 (Test-Convex).

1. *Repeat $2/\epsilon$ times:* Test-Interval($I_{0,n}$).
2. *If all of the tests in step 1 accepted, then accept; otherwise, reject.*

PROCEDURE 1 (Test-Interval($I_{i,j}$)).

1. *Check that $I_{i,j}$ is good with respect to $f$. If not, reject.*
2. *If $j - i > 7$, then: Uniformly at random call either* Test-Interval($I_{i,mid}$) *or* Test-Interval($I_{mid+1,j}$), *where $mid = \lfloor (i+j)/2 \rfloor$.*
3. *If the test in step 2 accepted, then accept; otherwise, reject.*

THEOREM 1. *If $f$ is convex, then Algorithm 1 always accepts, and if $f$ is $\epsilon$-far from convex, then the algorithm rejects with a probability of at least $2/3$.*

*Proof.* For the sake of brevity, unless stated otherwise, when we say that an interval is good, then we mean with respect to $f$. If $f$ is convex, then all intervals $I_{i,j}$ are good, and hence Algorithm 1 accepts with probability 1. In order to prove that if $f$ is $\epsilon$-far from convex, then the algorithm rejects with probability of at least $2/3$, we prove the contrapositive statement. Assume that the algorithm accepts with a probability greater than $1/3$. We will show that $f$ is $\epsilon$-close to a convex function.

To this end we define a tree whose vertices correspond to all possible intervals $I_{i,j}$ that may be tested recursively in calls to Test-Interval($I_{i,j}$). Specifically, the root of the tree corresponds to $I_{0,n}$. The children of the internal vertex corresponding to $I_{i,j}$ are the vertices corresponding to $I_{i,mid}$ and $I_{mid+1,j}$, where $mid = \lfloor (i+j)/2 \rfloor$. The leaves of the tree correspond to the smallest intervals tested, that is, intervals $I_{i,j}$ for which $j - i \leq 7$.

We say that an internal vertex in the tree is good if the corresponding interval is good. We say that a leaf is good if its corresponding interval and all its ancestors are good. Otherwise, the vertex (leaf) is bad. We say that a path from the root to a leaf is good if all vertices along it are good. Otherwise, the path is bad. For each level $\ell$ in the tree, $\ell = 0, \ldots, \log n$, let $\mathcal{B}_\ell$ be the subset of vertices in the $\ell$th level of the

tree that are bad but whose ancestors are all good. Let $\mathcal{B} = \bigcup_\ell \mathcal{B}_\ell$, and let $\epsilon_\ell$ be the fraction of vertices in level $\ell$ of the tree that belong to $\mathcal{B}_\ell$.

*Subclaim* 1. If Algorithm 1 accepts $f$ with a probability greater than $1/3$, then $\sum_\ell \epsilon_\ell \leq \epsilon$.

*Proof.* Assume by contradiction that $\sum_\ell \epsilon_\ell > \epsilon$. Observe that by the definition of $\mathcal{B}$, all leaves which are descendents of a vertex in $\mathcal{B}$ are bad, and every bad leaf either belongs to $\mathcal{B}$ or has a single ancestor in $\mathcal{B}$. Therefore, if $\sum_\ell \epsilon_\ell > \epsilon$, then the fraction of bad leaves is greater than $\epsilon$. But in such a case, the probability that the algorithm does not follow a bad path to a bad leaf (passing through a vertex in $\mathcal{B}$) in any one of its $2/\epsilon$ iterations is at most $(1-\epsilon)^{2/\epsilon} < e^{-2} < 1/3$. This contradicts our assumption that the algorithm accepts with a probability greater than $1/3$.

Hence we assume from now on that $\sum_\ell \epsilon_\ell \leq \epsilon$. Note also that in this case $I_{0,n} \notin \mathcal{B}$. We show how to modify $f$ in at most $\epsilon \cdot n$ places so that the resulting function, denoted $g$, is convex. In particular, we shall modify the value of $f$ on every bad interval $I_{i,j}$ whose corresponding vertex in the tree belongs to $\mathcal{B}$. The value of $g$ is defined to be the same as the value of $f$ on all points outside of these intervals. Since $\sum_\ell \epsilon_\ell \leq \epsilon$, the total fraction of points modified is at most $\epsilon$ as required. Observe that by the definition of the tree and $\mathcal{B}$, for every two intervals whose corresponding vertices belong to $\mathcal{B}$, the intersection of the intervals is empty. Hence we can modify each one of these intervals independently.

Let $I_{i,j}$ be a bad interval corresponding to a vertex in $\mathcal{B}$. We modify $f$ on points in $I_{i,j}$ as follows:

- $f(i), f(i+1)$, $f(j-1)$, and $f(j)$ remain unchanged. That is, set $g(i) = f(i)$, $g(i+1) = f(i+1)$, $g(j-1) = f(j-1)$, and $g(j) = f(j)$.
- For every $t$, $i+1 < t < j-1$, set $g(t) = f(i+1) + \frac{f(j-1)-f(i+1)}{(j-1)-(i+1)} \cdot (t-(i+1))$.

*Subclaim* 2. Let $I_{i,j}$ be a bad interval corresponding to a vertex in $\mathcal{B}$. Then for every $i < t < j$, $g(t) - g(t-1) \leq g(t+1) - g(t)$.

*Proof.* By definition of $\mathcal{B}$, the parent of $I_{i,j}$ is good (the parent exists by our assumption that $I_{0,n} \notin \mathcal{B}$). Hence

$$f(i+1) - f(i) \leq \frac{f(j-1) - f(i+1)}{(j-1)-(i+1)} \leq f(j) - f(j-1). \tag{3}$$

By definition of $g(\cdot)$, $g(i+1) - g(i) = f(i+1) - f(i)$, $g(j) - g(j-1) = f(j) - f(j-1)$, and for every $i+1 < t \leq j-1$, $g(t) - g(t-1) = \frac{f(j-1)-f(i+1)}{(j-1)-(i+1)}$. Therefore, for every $i+1 < t < j-1$, $g(t) - g(t-1) = g(t+1) - g(t)$, and for both $t = i+1$ and $t = j-1$, we have $g(t) - g(t-1) \leq g(t+1) - g(t)$ as required.

*Subclaim* 3. The function $g$ is convex.

*Proof.* We shall first show that all intervals $I_{i,j}$ corresponding to vertices in the tree are good with respect to $g$, and from this we derive the convexity of $g$.

We start with the first part. Consider any such interval $I_{i,j}$ whose corresponding vertex in the tree is $v$. Let $\mathit{Anchor} = \{i, i+1, \mathit{mid}-1, \mathit{mid}, \mathit{mid}+1, \mathit{mid}+2, j-1, j\}$ be the set of points which participate in the definition of a good interval $I_{i,j}$. We will show that the value of $g$ on points $p \in \mathit{Anchor}$ is such that the interval $I_{i,j}$ is good with respect to $g$. There are two cases:

1. The interval $I_{i,j}$ is good with respect to $f$, and $v$ does not have any ancestors in $\mathcal{B}$. If $v$ also has no descendents in $\mathcal{B}$, then it clearly remains good with respect to $g$, since no modification is performed on any point in the interval, and so $g(t) = f(t)$ for every $i \leq t \leq j$. Otherwise, $v$ has a descendent in $\mathcal{B}$. In this case, let $p \in \mathit{Anchor}$, let $v'$ be a descendent of $v$, and let $I_{i',j'}$ denote

the interval corresponding to $v'$. If $i' \le p \le j'$, then by definition of the tree, either $p = i'$ or $p = i' + 1$ or $p = j' - 1$ or $p = j'$. Therefore, even if $v' \in \mathcal{B}$ and the interval $I_{i',j'}$ is modified, then by the definition of $g$ we have that $g(p) = f(p)$ for every $p \in Anchor$. Thus $I_{i,j}$ remains good with respect to $g$.

2. Either $v \in \mathcal{B}$ or $v$ has an ancestor in $\mathcal{B}$. In the former case, let $v' = v$, and in the latter case let $v'$ be the ancestor that $v$ has in $\mathcal{B}$. Let $I_{i',j'}$ be the corresponding interval of $v'$. By definition, $I_{i,j} \subseteq I_{i',j'}$. By Subclaim 2, $g(t) - g(t-1) \le g(t+1) - g(t)$ for every $i' < t < j'$, and in particular for every $i < t < j$. It follows that $I_{i,j}$ is good with respect to $g$.

Hence all intervals corresponding to vertices in the tree are good with respect to $g$. We now prove that for every $0 < t < n$ it holds that $g(t) - g(t-1) \le g(t+1) - g(t)$, and thus $g$ is convex. Let $I_{i,j}$ be the smallest interval in the tree such that $i < t < j$. If $j - i \le 7$, then we are done, since the goodness of $I_{i,j}$ in this case means that $g$ is convex over the whole interval. Otherwise, either $t = mid$ or $t = mid + 1$, where $mid = \lfloor (i+j)/2 \rfloor$. To verify this, note that if this were not the case, then either $i < t < mid$ or $mid + 1 < t < j$. Hence $t$ is contained in a smaller interval in the tree, contradicting the minimality of $I_{i,j}$. But since $I_{i,j}$ is good with respect to $g$, $g(mid) - g(mid-1) \le g(mid+1) - g(mid)$, and $g(mid+1) - g(mid) \le g(mid+2) - g(mid+1)$. Thus we are done with the proof of Subclaim 3, and Theorem 1 follows. □

**3. Building blocks for our algorithms for testing inverse monge.** From this point on we focus on inverse Monge matrices. Analogous claims hold for Monge matrices. We also assume for simplicity that the dimensions of the matrices are $n_1 = n_2 = n$. In what follows we provide a characterization of inverse Monge matrices that is exploited by our algorithms. Given any real valued matrix $V = \{v_{i,j}\}_{i,j=0}^{i,j=n}$ we define an $(n+1) \times (n+1)$ matrix $C'_V = \{c_{i,j}\}_{i,j=0}^{i,j=n}$ as follows:

- $c_{0,0} = v_{0,0}$.
- For $i > 0$: $c_{i,0} = v_{i,0} - v_{i-1,0}$.
- For $j > 0$: $c_{0,j} = v_{0,j} - v_{0,j-1}$.
- And for every $i, j > 0$,

$$\begin{aligned} c_{i,j} &= (v_{i,j} - v_{i-1,j}) - (v_{i,j-1} - v_{i-1,j-1}) \\ &= (v_{i,j} - v_{i,j-1}) - (v_{i-1,j} - v_{i-1,j-1}). \end{aligned} \tag{4}$$

Let $C_V = \{c_{i,j}\}_{i,j=1}^{i,j=n}$ be the submatrix of $C'_V$ that includes all but the first (0th) row and column of $C'_V$. The following two claims are well known and easy to verify. We include their proofs for completeness.

CLAIM 2. *For every* $0 \le i, j \le n$, $v_{i,j} = \sum_{k=0}^{i} \sum_{\ell=0}^{j} c_{k,\ell}$.

*Proof.* The claim is proved by induction on $i$ and $j$.

The base case $i, j = 0$ holds by definition of $c_{0,0}$.

Consider any $i > 0$ and assume that the claim holds for every $k < i$, $j = 0$. We prove it for $i$ and for $j = 0$. By definition of $c_{i,0}$ we have $v_{i,0} = v_{i-1,0} + c_{i,0}$. By the induction hypothesis, $v_{i-1,0} = \sum_{k=0}^{i-1} c_{k,0}$, and the induction step follows. The claim is similarly proved for every $j > 0$ and $i = 0$.

Finally, consider any $i, j > 0$ and assume that the claim holds for every $k < i$ and $\ell \le j$, and for every $k \le i$ and $\ell < j$. We prove it for $i, j$. By definition of $c_{i,j}$,

$v_{i,j} = v_{i-1,j} + (v_{i,j-1} - v_{i-1,j-1}) + c_{i,j}$. By the induction hypothesis,

$$v_{i-1,j} + (v_{i,j-1} - v_{i-1,j-1}) = \sum_{k=0}^{i-1}\sum_{\ell=0}^{j} c_{k,\ell} + \sum_{\ell=0}^{j-1} c_{i,\ell},$$

and the induction step follows. ∎

CLAIM 3. *A matrix $V$ is an inverse Monge matrix if and only if $C_V$ is a nonnegative matrix.*

*Proof.* If $V$ is an inverse Monge matrix, then, in particular, for every $i, j \geq 1$ we have that $v_{i,j} + v_{i-1,j-1} \geq v_{i,j-1} + v_{i-1,j}$, which is equivalent to the condition $c_{i,j} \geq 0$.

In the other direction, consider any two points $(i, j)$ and $(i', j')$ such that $0 \leq i < i' \leq n$, $0 \leq j < j' \leq n$. Using Claim 2 we obtain

$$\begin{aligned} & v_{i',j'} - v_{i',j} - v_{i,j'} + v_{i,j} \\ &= \sum_{k=0}^{i'}\sum_{\ell=0}^{j'} c_{k,\ell} - \sum_{k=0}^{i'}\sum_{\ell=0}^{j} c_{k,\ell} - \sum_{k=0}^{i}\sum_{\ell=0}^{j'} c_{k,\ell} + \sum_{k=0}^{i}\sum_{\ell=0}^{j} c_{k,\ell} \\ &= \sum_{k=i+1}^{i'}\sum_{\ell=j+1}^{j'} c_{k,\ell}. \end{aligned} \tag{5}$$

But $C_V$ is nonnegative, and therefore $v_{i',j'} - v_{i',j} - v_{i,j'} + v_{i,j} \geq 0$ as required. ∎

It follows from Claim 3 that if we find some entry of $C_V$ that is negative, then we have evidence that $V$ is not an inverse Monge matrix. However, it is not necessarily true that if $V$ is far from being an inverse Monge matrix, then $C_V$ contains many negative entries. For example, suppose that $C_V$ is 1 in all entries except the entry $c_{n/2,n/2}$ which is $-n^2$. Then it can be verified that $V$ is very far from being an inverse Monge matrix (this can be proved by showing that there are $\Theta(n^2)$ disjoint quadruples $v_{i,j}, v_{i',j'}, v_{i,j'}, v_{i',j}$ in $V$ such that from any such quadruple at least one value should be changed in order to transform $V$ into an inverse Monge matrix). However, as our analysis will show, in such a case there are many submatrices in $C_V$ whose sum of elements is negative. Thus our testing algorithms will sample certain submatrices of $C_V$ and check that the sum of elements in each submatrix sampled is nonnegative. We first observe that it is possible to check this efficiently.

CLAIM 4. *Given access to $V$ it is possible to check in time $O(1)$ if the sum of elements in a given submatrix $A$ of $C_V$ is nonnegative. In particular, if the lower-left entry of $A$ is $(i, j)$ and its upper-right entry is $(i', j')$, then the sum of elements of $A$ is $v_{i',j'} - v_{i',j-1} - v_{i-1,j'} + v_{i-1,j-1}$.*

*Proof.* Assume that $A = (c_{k,\ell})_{k=i,\ell=j}^{k=i',\ell=j'}$ is a submatrix of $C_V$. Recall that for any $q, p$, we have $v_{q,p} = \sum_{k=0}^{q}\sum_{\ell=0}^{p} c_{k,\ell}$. Thus the sum of elements of $A$ is

$$\begin{aligned} \sum_{k=i}^{i'}\sum_{\ell=j}^{j'} c_{k,\ell} &= \sum_{k=0}^{i'}\sum_{\ell=j}^{j'} c_{k,\ell} - \sum_{k=0}^{i-1}\sum_{\ell=j}^{j'} c_{k,\ell} \\ &= \left(\sum_{k=0}^{i'}\sum_{\ell=0}^{j'} c_{k,\ell} - \sum_{k=0}^{i'}\sum_{\ell=0}^{j-1} c_{k,\ell}\right) - \left(\sum_{k=0}^{i-1}\sum_{\ell=0}^{j'} c_{k,\ell} - \sum_{k=0}^{i-1}\sum_{\ell=0}^{j-1} c_{k,\ell}\right) \\ &= (v_{i',j'} - v_{i',j-1}) - (v_{i-1,j'} - v_{i-1,j-1}) \,. \end{aligned}$$

Therefore computing the sum of elements of any submatrix $A$ of $C_V$ can be done by checking only four entries in the matrix $V$. ∎

**3.1. Filling submatrices.** An important building block for the analysis of our algorithms is a procedure for "filling in" a submatrix. That is, given constraints on the sum of elements in each row and column of a given submatrix, we are interested in assigning values to the entries of the submatrix so that these constraints are met.

Specifically, let $a_1, \ldots, a_s$ and $b_1, \ldots, b_t$ be nonnegative real numbers such that $\sum_{i=1}^{s} a_i \geq \sum_{j=1}^{t} b_j$. Then it is possible to construct an $s \times t$ nonnegative real matrix $T$ such that the sum of elements in column $j$ is exactly $b_j$ and the sum of elements in row $i$ is at most $a_i$. In the special case that $\sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j$, the sum of elements in row $i$ will equal $a_i$. In particular, this can be done by applying the following procedure, which is the same as the one applied to obtain an initial feasible solution for the linear-programming formulation of the transportation problem.

PROCEDURE 2 (fill matrix $T = (t_{i,j})_{i,j=1}^{i=s,j=t}$).
*Initialize* $\bar{a}_i = a_i$ *for* $i = 1, \ldots, s$ *and* $\bar{b}_j = b_j$ *for* $j = 1, \ldots, t$.
*(In each of the following iterations,* $\bar{a}_i$ *is an upper bound on what remains to be filled in row* $i$*, and* $\bar{b}_j$ *is what remains to be filled in column* $j$*.)*
*For* $j = 1,\ldots,t$*:*
  *For* $i = 1,\ldots,s$*:*
    *Assign to entry* $(i, j)$ *the value* $x = \min\{\bar{a}_i, \bar{b}_j\}$.
    *Update* $\bar{a}_i = \bar{a}_i - x$, $\bar{b}_j = \bar{b}_j - x$.

CLAIM 5. *Procedure 2 fills the matrix* $T$ *with nonnegative values* $t_{i,j}$ *such that at the end of the procedure,* $\sum_{i=1}^{s} t_{i,j} = b_j$ *for every* $j = 1, \ldots, t$*, and* $\sum_{j=1}^{t} t_{i,j} \leq a_i$ *for every* $i = 1, \ldots, s$*. If initially* $\sum_{j=1}^{t} b_j = \sum_{i=1}^{s} a_i$*, then* $\sum_{j=1}^{t} t_{i,j} = a_i$ *for every* $i = 1, \ldots, s$.

*Proof.* Notice that initially $\bar{a}_i = a_i \geq 0$ and $\bar{b}_j = b_j \geq 0$. Thus when we update $\bar{a}_i = \bar{a}_i - x = \bar{a}_i - \min\{\bar{a}_i, \bar{b}_j\} \geq 0$ and similarly $\bar{b}_j = \bar{b}_j - x = \bar{b}_j - \min\{\bar{a}_i, \bar{b}_j\} \geq 0$. Therefore the $\bar{a}_i$'s and $\bar{b}_j$'s are always nonnegative. Hence all values $x$ filled in $T$ are nonnegative, since $x = \min\{\bar{a}_i, \bar{b}_j\} \geq 0$. Furthermore, after each such update the new sum over the $\bar{a}_i$'s equals the old sum over the $\bar{a}_i$'s minus $x$, and a similar statement holds for the sum over the $\bar{b}_j$'s. Thus at all stages of the procedure, $\sum_{i=1}^{s} \bar{a}_i \geq \sum_{j=1}^{t} \bar{b}_j$, and if initially $\sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j$, then $\sum_{i=1}^{s} \bar{a}_i = \sum_{j=1}^{t} \bar{b}_j$.

We now show that the sum of elements in each column is as required. Observe that the procedure fills the columns one by one. Therefore when we start to fill column $j$ we have $\bar{b}_j = b_j$. Since $\sum_{i=1}^{s} \bar{a}_i \geq \sum_{j=1}^{t} \bar{b}_j$ at this stage, and all $\bar{a}_i$'s are nonnegative, necessarily, $\sum_{i=1}^{s} \bar{a}_i \geq \bar{b}_j = b_j$. Let $1 \leq k \leq s$ be the minimum integer such that $\sum_{i=1}^{k} \bar{a}_i \geq b_j$. Then by definition of the procedure, for every $i < k$, the entry $(i, j)$ is filled with the value $\bar{a}_i$, and the entry $(k + 1, j)$ is filled with the value $b_j - \sum_{i=1}^{k} \bar{a}_i$. The total is hence $b_j$ as required.

As for the rows, at all stages $\bar{a}_i$ equals $a_i$ minus the sum of all elements filled so far in row $i$. Therefore since $\bar{a}_i \geq 0$, then the sum of elements in row $i$ is at most $a_i$. Furthermore, if initially $\sum_{i=1}^{s} a_i = \sum_{j=1}^{t} b_j$, then the sum of elements in row $i$ will be exactly $a_i$. To show this note that at the end of the procedure, $\sum_{j=1}^{t} \bar{b}_j = 0$, since each $\bar{b}_j$ equals $b_j$ minus the sum of all elements in column $j$, and we have shown that the sum of elements in column $j$ is $b_j$. But $\sum_{i=1}^{s} \bar{a}_i = \sum_{j=1}^{t} \bar{b}_j$, and therefore also $\sum_{i=1}^{s} \bar{a}_i = 0$ at the end. Since $\bar{a}_i \geq 0$, this means that $\bar{a}_i = 0$. Hence the sum of elements in row $i$ must be $a_i$. ∎

**4. A testing algorithm for inverse monge matrices.** We first present a simple algorithm for testing if a matrix $V$ is an inverse Monge matrix whose running

time is $O(n/\epsilon)$. In the next section we show a significantly faster algorithm that is partly based on the ideas presented here. We may assume without loss of generality that $n$ is a power of 2. This is true since our algorithms probe the coefficients matrix $C_V$, and we may simply "pad" it by 0's to obtain rows and columns that have lengths which are powers of 2 and run the algorithm with $\epsilon \leftarrow \epsilon/4$. We shall need the following two definitions for both algorithms.

DEFINITION 4 (subrows, subcolumns, and submatrices). *A* subrow *in an $n \times n$ matrix is a consecutive sequence of entries that belong to the same row. The subrow $((i,j),(i,j+1),\ldots,(i,j+t-1))$ is denoted by $[\,]_{i,j}^{1,t}$. A* subcolumn *is defined analogously and is denoted by $[\,]_{i,j}^{s,1} = ((i,j),(i+1,j),\ldots,(i+s-1,j))$. More generally, an $s \times t$ submatrix whose bottom-left entry is $(i,j)$ is denoted $[\,]_{i,j}^{s,t}$.*

DEFINITION 5 (legal submatrices). *A subrow in an $n \times n$ matrix is a* legal *subrow if it can result from bisecting the row of length $n$ that contains it in a recursive manner. That is, a complete (length $n$) row is legal, and if $[\,]_{i,j}^{1,t}$ is legal, then so are $[\,]_{i,j}^{1,t/2}$ and $[\,]_{i,j+t/2}^{1,t/2}$. A legal subcolumn is defined analogously. A submatrix is* legal *if both its rows and its columns are legal.*

Note that the legality of a subrow $[\,]_{i,j}^{1,t}$ is not dependent on the actual row $i$ it belongs to, but rather it depends on its starting position $j$ and ending position $j+t-1$ within its row. An analogous statement holds for legal subcolumns. See also Figure 1 for an illustration of the concept of legal submatrices.

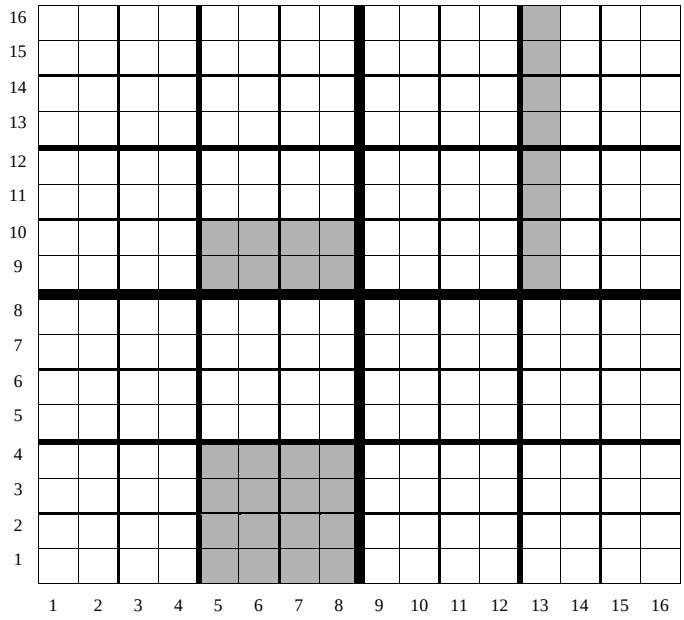


FIG. 1. *An illustration of three legal submatrices. One of the legal submatrices is a square ($4 \times 4$) submatrix, and the other two are rectangular (but legal) submatrices. The $8 \times 1$ submatrix on the top-right is a legal subcolumn.*

Although a submatrix is just a collection of positions (entries) in an $n \times n$ matrix, we talk throughout the paper about sums of elements in certain submatrices $A$ of $C_V$. In this we mean the sum of elements of $C_V$ determined by the set of positions in $A$.

DEFINITION 6 (good and bad submatrices). *We say that a submatrix $A$ of $C_V$ is*

good *if the sum of elements in each row of $A$ is nonnegative and the sum of elements in each column of $A$ is nonnegative. Otherwise, $A$ is* bad.

DEFINITION 7 (good and bad points). *We say that point $(i, j)$ is* good *if all legal* square *submatrices $A$ of $C_V$ which contain $(i, j)$ are good. Otherwise, the point is* bad.

ALGORITHM 2 (Test-Monge I).

1. *Choose $8/\epsilon$ points in the matrix $C_V$ and check that they are good.*
2. *If all points are good, then accept; otherwise, reject.*

By Claim 4, it is possible to check in constant time that the sum of elements in a subrow (subcolumn) of $C_V$ is nonnegative. Therefore, it is possible to test that an $s \times s$ square submatrix $A$ of $C_V$ is good in time $\Theta(s)$. Notice that every point in an $n \times n$ matrix is contained in $\log n$ legal square submatrices. Hence the time required to check whether a point is good is $O(n)+O(n/2)+\cdots+O(n/2^i)+\cdots+O(1) = O(n)$, and the complexity of the algorithm is $O(n/\epsilon)$.

THEOREM 2. *If $V$ is an inverse Monge matrix, then Algorithm* 2 *always accepts, and if $V$ is $\epsilon$-far from being an inverse Monge matrix, then Algorithm* 2 *rejects with probability at least* $2/3$.

*Proof.* The first part of the theorem follows directly from Claim 3. In order to prove the second part of the theorem, we show that if $V$ is $\epsilon$-far from being inverse Monge, then $C_V$ contains more than $(\epsilon/4)n^2$ bad points. The second part of the theorem directly follows because the probability in such a case that no bad point is selected by the algorithm is at most $(1-\epsilon/4)^{(8/\epsilon)} < e^{-2} < 1/3$.

Assume contrary to the claim that $C_V$ contains at most $(\epsilon/4)n^2$ bad points. We shall show that by modifying at most $\epsilon n^2$ entries in $V$ we obtain an inverse Monge matrix (in contradiction to our assumption concerning $V$). Let us look at the set of bad points in $C_V$, and for each such bad point look at the largest bad legal square submatrix in $C_V$ that contains this bad point. By our assumption on the number of bad points, it must be the case that the area of all these maximal bad submatrices is at most $(\epsilon/4)n^2$, because all the points in a bad submatrix are bad.

For each maximal bad legal square submatrix $B$ of $C_V$ we will look at the legal square submatrix $A$ that contains $B$. By definition of legal square submatrices, the matrix $A$ is uniquely defined. By the maximality of $B$, the submatrix $A$ must be good. Indeed, since $B$ is maximal, if it is of size $s \times s$, where $s < n$, then the legal square submatrix of size $2s \times 2s$ that contains it must be good. But if $s = n$, then $B = C_V$, implying that all $n^2$ points in $C_V$ are bad, contradicting our assumption on the number of bad points.

Next observe that every two different maximal bad legal square submatrices $B$ and $B'$ are disjoint. This is true since every two different legal square submatrices are either disjoint or one is contained in the other. Combining this with the fact that for each maximal bad legal square submatrix we take the good square legal submatrix that is four times its size, the area of the union of all these good submatrices is at most $4 \cdot (\epsilon/4)n^2 = \epsilon n^2$.

Turning to the collection of resulting good submatrices, note that every two of these submatrices are either disjoint, or are exactly the same, or one is contained in the other. If a good submatrix is strictly contained in another one, then we ignore it and deal only with the larger good submatrix containing it. Thus we have a set of disjoint good submatrices that contain all negative entries in the matrix. For each of these good submatrices $A$, we modify $A$ so that it contains only nonnegative elements, and the sum of elements in each row and column of $A$ remains as it was. This can

be done by applying Procedure 2 to $A$ as described in section 3.1 (using the actual (nonnegative) sums of rows and columns of $A$ as the input to the procedure).

Note that after modifying all these good submatrices of $C_V$, the new matrix $C_V$ is nonnegative, and thus the corresponding new matrix $V$ must be an inverse Monge matrix. It remains to show that at most $\epsilon n^2$ values were changed in $V$ following the changes to $C_V$. Notice that we made sure that the sum of elements in each row and column of each modified submatrix $A$ remains as it was. Therefore the values of all points $v_{k,\ell}$ in $V$ that are outside $A$ are not affected by the change to $A$, since by Claim 2 we have that $v_{k,\ell} = \sum_{i=0}^{k} \sum_{j=0}^{\ell} c_{i,j}$. □

**5. A faster algorithm for inverse monge matrices.** Algorithm 2 described above has running time linear in $n$, which is already sublinear in the size of the matrix, $n^2$. In this section we show how to significantly improve the dependence on $n$. We present a variant of the algorithm whose running time is $O(\epsilon^{-1} \log^2 n)$. The new algorithm will be based on a similar principle as that of Algorithm 2. That is, it will uniformly select points and verify that certain submatrices that contain them are good. However, there will be two main differences which we now describe briefly.

Algorithm 2 suffers from a relatively slow running time, since for each submatrix that the algorithm checks, it verifies that the sum of elements in *every* row and column is nonnegative. Therefore, we first relax the concept of a good submatrix and demand only that the sum of *all* its elements be nonnegative (instead of the sum of every row and column). This change, however, requires us to check for each point selected by the algorithm, not only that the legal *square* submatrices which contain it are good, but rather to verify that *all* legal submatrices that contain the point are good. Actually, we check something slightly stronger: The algorithm will verify for each legal submatrix $T$ that it examines that the four legal equal-size submatrices that reside within $T$ and are half of $T$'s length in each dimension are good as well. In order to formalize the above, we first redefine the concepts of good (bad) submatrices and good (bad) points, and introduce the notion of tainted submatrices and tainted points.

DEFINITION 8 (good and bad submatrices and points). *A (legal) submatrix $T$ of $C_V$ is* good *if the sum of all its elements is nonnegative. Otherwise, $T$ is* bad.

*A point is* good *if every legal submatrix of $C_V$ that contains it is good. Otherwise, the point is* bad.

DEFINITION 9 (tainted submatrices and points). *A good legal submatrix $T$ of $C_V$ is* tainted *if any one of the four legal submatrices that it contains and that are half its height and half its width is bad. A point is* tainted *if some legal submatrix that contains it is tainted.*

Note that every bad point is tainted, but good points may be tainted as well.

For the sake of the presentation, we shall assume that every row and every column in $C_V$ (that is, every subrow and subcolumn of length $n$) have nonnegative sums. In subsection 5.2 we explain how to remove this assumption. Note that this assumption implies that every $s \times n$ submatrix is good, and similarly every $n \times s$ submatrix is good (but of course it has no implications on smaller submatrices).

ALGORITHM 3 (Test-Monge II).

1. *Uniformly select $2/\epsilon$ points in the matrix $C_V$ and check for each of them whether it is tainted.*
2. *If no point selected is tainted, then accept; otherwise, reject.*

Note that by Definition 5, each point in an $n \times n$ matrix is contained in $O(\log^2 n)$ legal submatrices. Thus by Claim 4, checking whether a point is tainted takes time

$O(\log^2 n)$. Therefore the running time of the algorithm is $O((\log^2 n)/\epsilon)$.

THEOREM 3. *If $V$ is an inverse Monge matrix, then Algorithm* 3 *always accepts, and if $V$ is $\epsilon$-far from being an inverse Monge matrix, then Algorithm* 3 *rejects with probability at least* 2/3.

**5.1. Outline of the proof of Theorem 3.** If $V$ is an inverse Monge matrix, then by Claim 3 all elements in $C_V$ are nonnegative. This directly implies that all (legal) submatrices are good, and so all points are good and are not tainted. Hence in this case the algorithm always accepts. Suppose that $V$ is $\epsilon$-far from being inverse Monge. We claim that in such a case $C_V$ must contain more than $\epsilon n^2$ tainted points, causing the algorithm to reject with probability at least

$$1 - (1-\epsilon)^{(2/\epsilon)} > 1 - e^{-2} > 2/3.$$

Assume contrary to the claim that $C_V$ contains at most $\epsilon n^2$ tainted points. Our goal from this point on is to show that in such a case $V$ is $\epsilon$-close to being an inverse Monge matrix.

The proof of this part will follow along similar lines to those used in the proof of Theorem 2. That is, we consider all maximal bad legal submatrices of $C_V$, and for each such bad submatrix we consider the legal good submatrix that is four times its area and contains it. Once again, this submatrix is unique. By Definition 9, this submatrix is tainted. We then take the union of all these good but tainted submatrices. By our assumption on the number of tainted points, the area of this union is at most $\epsilon n^2$ since all points in the union are tainted.

Finally, we show how to modify the values in this union so that the resulting matrix is an inverse Monge matrix. This time, however, since the maximal bad submatrices may intersect (which was not the case in the slower algorithm), the good tainted submatrices that contain them may intersect in nontrivial ways (that is, not only by coinciding or by strict containment). As a result, the union of the good submatrices has a possibly complex structure (and in particular it is no longer a simple union of disjoint submatrices), and the process of properly modifying this union is much more involved. We now describe precisely the necessary definitions and proceed with a detailed proof.

DEFINITION 10 (maximal bad legal submatrix). *A bad legal submatrix $T$ of $C_V$ is a* maximal bad legal submatrix *of $C_V$ if it is not contained in any larger bad legal submatrix of $C_V$.*

Now consider all maximal bad legal submatrices of $C_V$. Note that every negative entry in $C_V$ is contained in the union of these bad submatrices. For each such submatrix $B$ let us take the (unique) legal submatrix $T$ that contains it and has twice the number of rows and twice the number of columns of $B$ (by our assumption that all full rows and columns have a nonnegative sum it is indeed possible to double the rows and columns of $B$). Then by the maximality of $B$, the resulting submatrix is good. We now take the union of all these good (but tainted) legal submatrices. Recall that the area of the union of all tainted (legal) submatrices of $C_V$ is at most $\epsilon n^2$. Denote the union of all these good tainted submatrices by $R$. See, for example, Figure 2.

In subsections 5.3 and 5.4 we show that it is possible to change the (at most $\epsilon n^2$) entries of $C_V$ within $R$ to nonnegative values so that the following property holds.

PROPERTY 1 (sum property for $R$). *For every point $(i,j)$ outside of $R$, the sum of the elements in the modified entries $(i',j')$ within $R$ such that $i' \le i$ and $j' \le j$ is the same as in the original matrix $C_V$.*

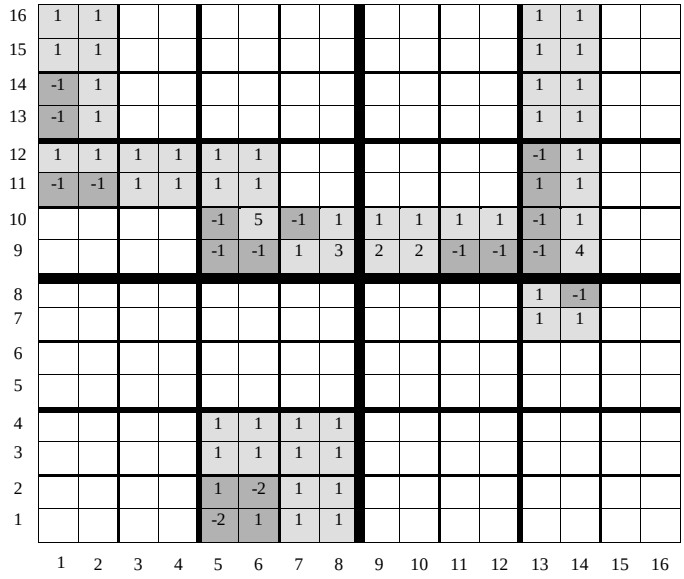


FIG. 2. *An example of the structure of a subset $R$, where $R$ is the union of all gray cells in the matrix (both dark and light gray). All values in cells outside of $R$ are nonnegative and are not displayed for the sake of simplicity. The bad legal submatrices determining $R$ are the dark gray submatrices. Each is contained inside a good but tainted legal submatrix that has twice the number of rows and twice the number of columns (good tainted submatrices are marked both by light and dark gray). For example, there is a bad submatrix in column* 1*, rows* 13 *and* 14*, and the good legal submatrix containing it is the submatrix over columns* 1 *and* 2 *and rows* 13 *through* 16*. Observe that maximal bad legal submatrices may intersect. For example, the bad submatrix containing the two cells in row* 9 *and columns* 5 *and* 6 *intersects with the bad submatrix containing the two cells in column* 5 *and rows* 9 *and* 10*. Their corresponding good submatrices also intersect.*

Let $\tilde{C}_V$ be the matrix obtained from $C_V$ by modifying $R$ so that Property 1 holds, and let $\tilde{V}$ be the matrix which corresponds to $\tilde{C}_V$. Then it follows from Claim 2 that $\tilde{V}$ is at most $\epsilon$-far from the original matrix $V$, and this completes the proof of Theorem 3. Before we continue with showing how to obtain Property 1, we explain shortly how to remove the assumption that all (full) rows and columns in $C_V$ have a nonnegative sum.

**5.2. Dealing with rows/columns having a negative sum.** Suppose first that $\epsilon \leq 4/n$. Then we may directly check in time $O(1/\epsilon)$ that in fact all rows and columns of the matrix $C_V$ have nonnegative sums (using Claim 4) and reject if some row or column has a negative sum. Hence in this case our assumption is valid. Thus assume that $\epsilon > 4/n$.

First we slightly modify Algorithm 3 so that it uniformly selects $4/\epsilon$ points in $C_V$ (instead of $2/\epsilon$). In such a case, if $C_V$ contains more than $(\epsilon/2)n^2$ tainted points, then the algorithm rejects with probability at least $2/3$. We thus assume that $C_V$ contains at most $(\epsilon/2)n^2$ tainted points and strive to show that in such a case $V$ is $\epsilon$-close to being an inverse Monge matrix. Since we do not assume that every row and column in $C_V$ has a nonnegative sum, we first modify $C_V$ so that it has this property.

Consider each row $i$ in $C_V$ whose sum of elements in negative. Suppose that we modify the last entry in the row, $c_{i,n}$, so that the new sum of all elements is 0. Similarly, we modify the last entry $c_{n,j}$ in each column $j$ that has a negative sum. Let $\bar{C}_V$ be the resulting matrix, and let $\bar{V}$ be the matrix corresponding to $\bar{C}_V$. Then

all rows and columns in $\bar{C}_V$ have a nonnegative sum, and by Claim 2 $\bar{V}$ and $V$ differ on at most $2n-1 < (\epsilon/2)n^2$ entries (at most all elements in the last column and last row).

Now we may define the region $R$ as we did in the previous subsection. Note that in this case the area of the region $R$ is at most $(\epsilon/2)n^2$. We can therefore continue in proving that it is possible to modify only the elements within $R$ so that they are all nonnegative and Property 1 holds. This will imply that the total number of entries that should be modified (first to obtain nonnegative rows and columns, and then to refill $R$) is at most $\epsilon n^2$, as desired.

**5.3. Refilling $R$ to obtain Property 1.** Let $R$ be as defined in section 5.1. Recall that $R$ consists of a union of good legal submatrices. (The fact that they are tainted is no longer relevant.) In the following discussion, when we talk about elements in submatrices of $R$ we mean the elements in $C_V$ determined by the corresponding set of positions in $R$.

We are interested in refilling the entries in $R$ with *nonnegative* values so that Property 1 will hold. Note that if $R$ is just a submatrix (block) of $C_V$, then we can use Procedure 2 to refill $R$ as desired. However, in general the structure of $R$ is more complex. We show that there is a way to partition $R$ into disjoint blocks and refill each block using Procedure 2. In subsection 5.3.1 we define precisely what blocks are and present several other notions that are needed for the refilling procedure. The refilling procedure for $R$ is described in subsection 5.3.2, and its correctness is proved in subsection 5.4.

**5.3.1. Preliminaries for the refilling procedure.** As stated above, the refilling procedure will partition $R$ into disjoint blocks (submatrices) and fill each block separately with nonnegative values so that Property 1 is maintained. We start with defining the following term that will be needed to define blocks.

DEFINITION 11 (maximal (legal) subrow/column). *Given a subset $R$ of entries in an $n \times n$ matrix, a subrow $T$ is a* maximal (legal) subrow with respect to $R$ *if $T$ is contained in $R$ and there is no larger (legal) subrow $T'$ such that $T \subset T' \subseteq R$. A* maximal (legal) subcolumn with respect to $R$ *is defined analogously.*

For the sake of succinctness, whenever it is clear what $R$ is, we shall just say maximal (legal) subrow and drop the suffix "with respect to $R$." Note that a maximal subrow is simply a maximal consecutive sequence of entries in $R$ that belong to the same row, while a maximal legal subrow is a more constrained notion. In particular, a maximal subrow may be a concatenation of several maximal legal subrows. We can now define blocks as follows.

DEFINITION 12 (maximal block). *A maximal block $B = [\,]_{i,j}^{s,t}$ in $R$ is a submatrix contained in $R$ which has the following property: It consists of a maximal consecutive sequence of maximal legal subcolumns of the same height. The maximality of each subcolumn is as in Definition* 11. *That is, for every $j \le r \le j+t-1$, the column $[\,]_{i,r}^{s,1}$ is a maximal legal subcolumn (with respect to $R$).*

*The* height *of a maximal block $B$ is the height of the columns in $B$ (equivalently, the number of rows in $B$).*

The maximality of the sequence of subcolumns in a block $B = [\,]_{i,j}^{s,t}$ means that we can extend the sequence of columns neither to the left nor to the right. That is, neither $[\,]_{i,j-1}^{s,1}$ nor $[\,]_{i,j+t}^{s,1}$ is a maximal legal subcolumn in $R$. (Specifically, each either is not fully contained in $R$ or $R$ contains a larger legal subcolumn that contains it.)

We shall sometimes refer to maximal blocks simply as blocks. Observe that by this definition, $R$ is indeed partitioned in a unique way into maximal disjoint blocks.

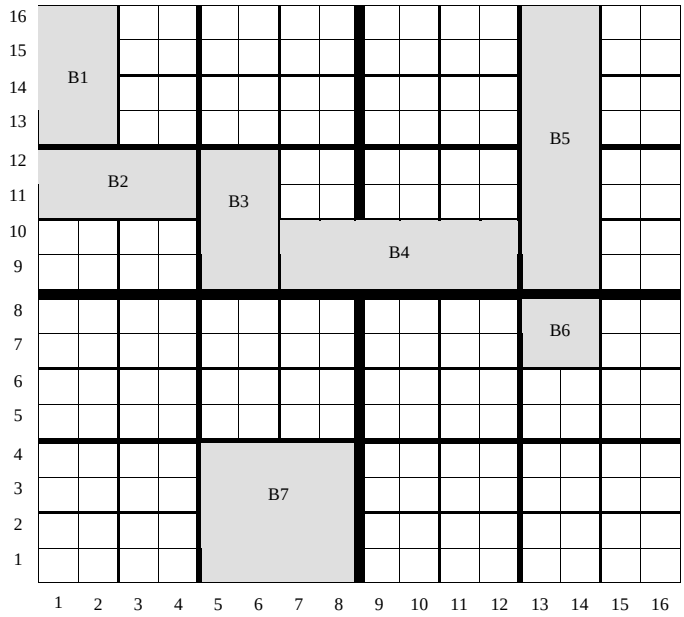


FIG. 3. *An example of the partition of $R$ shown in Figure* 2 *into maximal blocks (numbered $B_1$–$B_7$). Note that the ratio between the heights of any two blocks is always a power of* 2. *Furthermore, the blocks are aligned in the following way. Suppose a block $B$ has height $s$, and a block $B'$ has height $s' \leq s$ and some of their subrows belong to the same row of the matrix (e.g., $B_3$ and $B_4$, or $B_4$ and $B_5$). Then the shorter block $B'$ must be aligned with either the first or second half of $B$, or with one of the quarters of $B$, or with one of its eighths, etc.*

See Figure 3 for an illustration to how the subset $R$ from Figure 2 is partitioned into maximal blocks.

Three additional notions that will be needed for the refilling procedure are defined below. The first two are illustrated in Figure 4.

DEFINITION 13 (covers). *We say that a submatrix $A$* covers *a given block $B$ with respect to $R$ if $B \subseteq A \subseteq R$ and the number of rows in $A$ equals the height of $B$.*

*We say that $A$ is a* maximal row-cover *with respect to $R$ if $A$ consists of maximal subrows with respect to $R$.*

DEFINITION 14 (borders). *We say that a submatrix $T = [\ ]_{i,j}^{s,t}$* borders *another submatrix $T' = [\ ]_{i',j'}^{s',t'}$ if $i' \leq i + s - 1$ and $i \leq i' + s' - 1$, and either $j' = j + t$ (so that $T$ is to the left of $T'$) or $j' + t' = j$ (so that $T$ is to the right of $T'$).*

DEFINITION 15 (sums). *For a given submatrix $T$, we denote the sum of the elements in $T$ by $sum(T)$.*

**5.3.2. The procedure for refilling *R*.** We now describe the procedure that refills the entries of $R$ with nonnegative values so as to obtain Property 1. Recall that $R$ is a disjoint union of maximal blocks. Hence if we remove a maximal block from $R$, then the maximal blocks of the remaining structure are simply the remaining maximal blocks of $R$. For simplicity of this introductory discussion, after removing a block from $R$, we refer to the remaining structure as $R$. The procedure described below will remove the blocks of $R$ one by one, in order of increasing (nondecreasing) height, and refill each block separately using Procedure 2.

Recall that when (re)filling an $s \times t$ submatrix, Procedure 2 is provided with non-

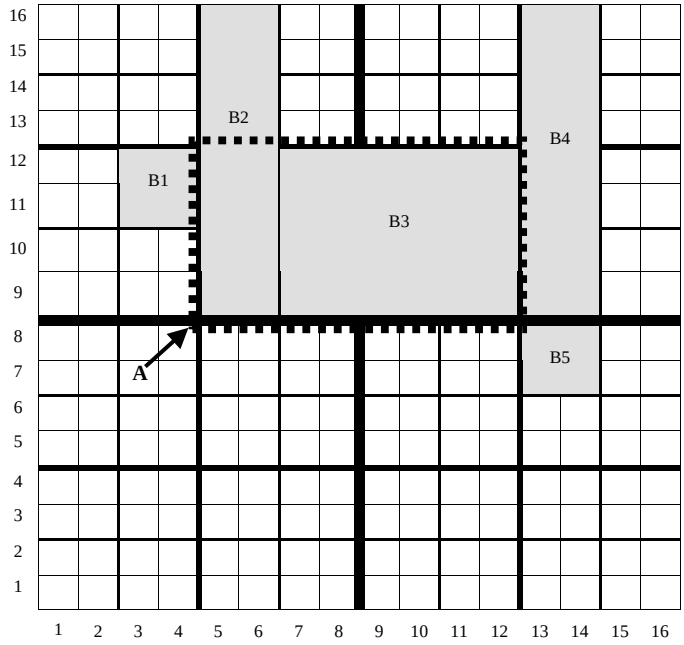


FIG. 4. *An illustration of the notions of covers and borders. Here the submatrix A (extending from row* 9 *to* 12 *and from column* 5 *to* 12*) covers the block* $B_3$ *(but is not a maximal row-cover with respect to R). The submatrix A borders block* $B_1$ *(from the left of A) and block* $B_4$ *(from the right of A).*

negative values $a_1, \ldots, a_s$ and $b_1, \ldots, b_t$ such that $\sum_{i=1}^{s} a_i \geq \sum_{j=1}^{t} b_j$. It then fills the submatrix with nonnegative values so that the sum of elements in column $j$ is exactly $b_j$ and the sum of elements in row $i$ is at most $a_i$. Whenever we apply Procedure 2 to a block $B$, the column sums $b_1, \ldots, b_t$ are simply set to be the sums of the elements in the corresponding subcolumns of $B$ in $C_V$. By definition of (maximal) blocks, these subcolumns are maximal legal subcolumns, and as we show in subsection 5.4.1, this ensures that their sums are nonnegative.

The setting of the upper bounds $a_1, \ldots, a_s$ for the row sums is a little more involved. At any point in the algorithm, each maximal subrow $L$ is associated with a *designated* sum, denoted $\overline{sum}(L)$. This is the sum we intend it to have when the refilling procedure terminates. Initially, for every maximal subrow $L$ in $R$, we set $\overline{sum}(L) = sum(L)$. That is, $\overline{sum}(L)$ is equal to the original sum of subrow $L$ in $C_V$. In subsection 5.4.1 we show that these sums are all nonnegative. When refilling a block $B$, we first find the row-cover $A$ of $B$ that is a maximal row-cover with respect to (the current) $R$. Since the blocks are filled by order of height and blocks are removed after they are filled, such a maximal row-cover must exist when $B$ is covered and is unique. We then use the designated sums of the (maximal) rows of $A$ as the upper bounds $a_1, \ldots, a_s$ for the sums of rows of $B$. As we prove subsequently, it always holds that $\sum_{i=1}^{s} a_i \geq \sum_{j=1}^{t} b_j$ as required by Procedure 2. After removing a block $B$ from $R$, we obtain new, shorter, maximal subrows in the remaining structure $R \setminus B$, and we must associate with these shorter subrows new designated sums. Procedure 2 is used here as well to determine how to set these designated row sums, in a manner explained in detail in step 3 below. For an illustration, see Figure 5.

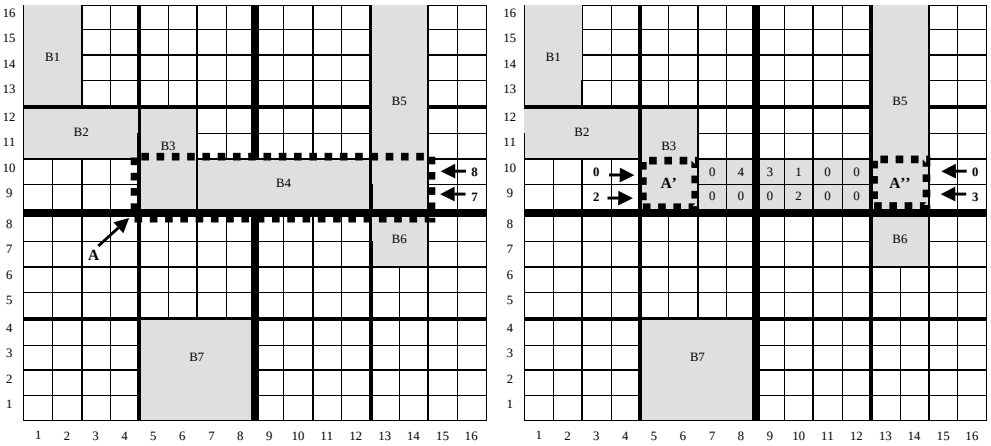


FIG. 5. *An illustration of one iteration of step* 3 *in Procedure* 3*, where we apply the procedure to the matrix illustrated in Figures* 2 *and* 3*. The first block filled may be either* $B_2$*,* $B_4$*, or* $B_6$ *(all three have height* 2*, which is the minimum among all blocks). Here we have selected to refill* $B_4$ *first. On the left we see the maximal row-cover* $A$ *that covers* $B_4$*, where the designated sums of the two rows of* $A$ *are* 8 *and* 7 *(in accordance with the values appearing in Figure* 2*). On the right we see the values that the Procedure* 2 *has entered in the cells of* $B_4$*. We also see the two submatrices,* $A'$ *and* $A''$*, that remain of* $A$ *after* $B_4$ *is removed from* $R$ *and the designated sums of the new maximal rows in* $A'$ *and* $A''$*.*

PROCEDURE 3 (refill $R$).

1. *We assign each maximal subrow* $L$ *in* $R$ *a designated sum of elements for that row, which is denoted by* $\overline{sum}(L)$*. Initially, we set* $\overline{sum}(L)$ *to be* $sum(L)$*.*
2. *Let* $m$ *be the number of maximal blocks in* $R$*, and let* $R_1 = R$*.*
3. *For* $p = 1, \ldots, m$ *we do the following:*
   (a) *Let* $B_p$ *be a maximal block in* $R_p$ *whose height is minimum among all maximal blocks of* $R_p$*, and assume that* $B_p$ *is an* $s \times t$ *submatrix. Let* $A_p$ *be a maximal row-cover of* $B_p$ *with respect to* $R_p$*. For* $1 \le \ell \le s$*, let* $L_\ell$ *denote the subrow of* $A_p$ *that covers the* $\ell$*th subrow of* $B_p$*.*
   (b) *Refill* $B_p$ *by applying Procedure* 2 *(see section* 3.1*), where the sum filled in the* $k$*th subcolumn of* $B_p$*,* $1 \le k \le t$*, should be the original sum of this subcolumn in* $C_V$*, and the sum filled in the* $\ell$*th subrow of* $B_p$*,* $1 \le \ell \le s$*, is at most* $\overline{sum}(L_\ell)$*.*
   *For each* $1 \le \ell \le s$*, let* $x_\ell$ *denote the sum of elements filled by Procedure* 2 *in the* $\ell$*th subrow of* $B_p$*.*
   (c) *Let* $R_{p+1} = R_p \setminus B_p$*. We next assign designated sums to the rows of* $R_{p+1}$ *that have been either shortened or broken into two parts by the removal of* $B_p$ *from* $R_p$*. This is done as follows:*
   *The set* $A_p \setminus B_p$ *is the union of two nonconsecutive submatrices,* $A'$ *and* $A''$*, so that* $A'$ *borders* $B_p$ *from the left of* $B_p$ *and* $A''$ *borders* $B_p$ *from the right of* $B_p$ *(where it is possible that one or both of these submatrices does not exist). Let* $L'_\ell$ *and* $L''_\ell$ *be the subrows in* $A'$ *and* $A''$*, respectively, that are contained in subrow* $L_\ell$ *of* $A_p$*. We assign to* $L'_\ell$ *and* $L''_\ell$ *nonnegative designated sums,* $\overline{sum}(L'_\ell)$ *and* $\overline{sum}(L''_\ell)$*, that satisfy the following:*

$$\overline{sum}(L'_\ell) + \overline{sum}(L''_\ell) = \overline{sum}(L_\ell) - x_\ell$$

*and, furthermore,*

$$\sum_{row\ L \in A'} \overline{sum}(L) = sum(A'), \qquad \sum_{row\ L \in A''} \overline{sum}(L) = sum(A'').$$

*This is done by applying Procedure* 2 *to a* $2 \times s$ *matrix whose sums of columns are* $sum(A')$ *and* $sum(A'')$ *and sums of rows are* $\overline{sum}(L_\ell) - x_\ell$, *where* $1 \le \ell \le s$.
*(Note that one or both of* $A'$ *and* $A''$ *may not exist. This can happen if* $B_p$ *bordered* $A_p \setminus B_p$ *on one side and its boundary coincided with* $R_p$ *or if* $A_p = B_p$. *In this case, if, for example,* $A'$ *does not exist, then we view it as a submatrix of height* 0, *where* $sum(A') = 0$.*)*

**5.4. Proving that Procedure 3 is correct.** In order to prove that Procedure 3 is correct we have to prove two claims. First, we have to show that the procedure does not "get stuck," namely, that all iterations of the procedure can be completed. Second, we have to prove that at the end of the procedure, the refilled structure $R$ has Property 1. Before we prove these two claims we first prove some properties relating to the sum of elements in maximal blocks and other submatrices of $R$. These properties will be used to show that the procedure does not get stuck.

**5.4.1. Sums of blocks and other submatrices.** We first prove the following simple lemma regarding the sum of elements in maximal legal subrows and subcolumns of $R$.

LEMMA 6. *The sum of elements in every maximal legal subrow and every maximal legal subcolumn in* $R$ *is nonnegative.*

*Proof.* We prove the lemma for maximal legal subrows. The claim for maximal legal subcolumns is analogous. Assume, contrary to the claim, that $R$ contains some maximal legal subrow $L = [\ ]_{i,j}^{1,t}$ whose sum of elements is negative. Let $T$ be the maximal bad legal submatrix in $C_V$ that contains $L$. By the maximality of $L$, necessarily $T = [\ ]_{i',j}^{s,t}$ for some $i' \le i$ and $s \ge 1$. That is, the rows of $T$ (one of which is $L$) are of length $t$. By the construction of $R$, $R$ must contain a good legal submatrix $T'$ that contains $T$ and is twice as large in each dimension. But this contradicts the maximality of $L$. □

It directly follows from Lemma 6 that every maximal row in $R$ has a nonnegative sum and that every maximal block has a nonnegative sum. We would like to characterize other submatrices of $R$ whose sum is necessarily nonnegative.

LEMMA 7. *Consider any two maximal blocks* $B = [\ ]_{i,j}^{s,t}$ *and* $B' = [\ ]_{i',j'}^{s',t'}$, *where* $i \le i' \le i+s-1$, $i'+s' \le i+s$. *That is,* $B$ *has height* $s$ *and* $B'$ *has height* $s' \le s$, *and* $B'$ *starts at row* $i' \ge i$ *and ends at row* $i'+s'-1 \le i+s-1$. *Consider the submatrix* $T$ *of height* $s$ *"between them." That is,* $T = [\ ]_{i,j+t}^{s,j'-(j+t)}$ *or* $T = [\ ]_{i,j'+t'}^{s,j-(j'+t')}$. *Suppose that* $T \subset R$. *Then* $sum(T) \ge 0$.

See Figure 6 for a illustration of the lemma and its proof.

*Proof.* Assume without loss of generality that $B'$ is to the right of $B$ (that is, $j' \ge j+t$ and $T = [\ ]_{i,j+t}^{s,j'-(j+t)}$). If $T$ is empty, then the claim follows trivially since $sum(T) = 0$. Hence we may assume from now on that $T$ is not empty, and we separate the proof into two cases.

*Case* 1. $T$ *is a legal submatrix.* Assume, contrary to the claim, that $sum(T) < 0$. That is, $T$ is a bad legal submatrix. Let $T'$ be the maximal bad legal submatrix containing $T$ (where $T'$ may equal $T$). By construction of $R$, $R$ should contain a good legal submatrix $T''$ that contains $T'$ and has twice the number of rows and twice the

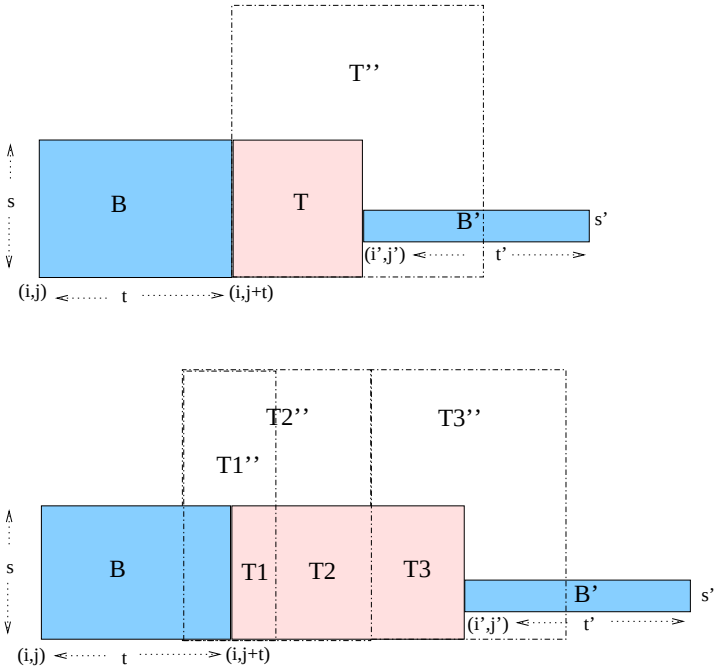


FIG. 6. *An illustration for Lemma* 7. *The figure on the top illustrates the case in the proof of Lemma* 7, *where* $T$ *is a legal submatrix (for simplicity, we assume* $T' = T$*). The figure on the bottom illustrates the second case in the proof when* $T$ *is a union of legal submatrices (all having the height of* $B$*).*

number of columns. But this would contradict the maximality of the subcolumns of $B$ or of $B'$. To see why this is true, assume without loss of generality that for any legal subcolumn $[\,]^{s,1}_{i,r}$, the legal column that is twice its height is $[\,]^{2s,1}_{i,r}$ (the case in which it is $[\,]^{2s,1}_{i-s,r}$, is treated analogously). Then $T''$ must contain either the subcolumn $[\,]^{2s,1}_{i,j'}$ or the subcolumn $[\,]^{2s,1}_{i,j+t-1}$ (depending on the identity of the legal subrows that are twice the length of the rows of $T$). In the first case we would get a contradiction to the fact that $B'$ is a maximal block, and in the second case we would get a contradiction to the fact that $B$ is a maximal block.

*Case* 2. $T$ *is not a legal submatrix.* Observe that its columns are necessarily legal subcolumns (given that the columns of $B$ are legal). Hence, only its rows are not legal subrows. Therefore, $T$ can be partitioned into submatrices $T_1, \ldots, T_k$ such that each is of height $s$ and is a maximal legal submatrix *with respect to* $T$. We claim that for every $T_\ell$, $sum(T_\ell) \geq 0$. Consider any fixed $T_\ell$. By its maximality with respect to $T$, we know that the legal subrows that contain the rows of $T_\ell$ and are twice their length are not strictly contained in $T$, but rather they extend either to the right or to the left of $T$. Hence these rows (or some of them in case the height of $B'$ is strictly smaller than the height of $T_\ell$) must intersect either $B$ or $B'$. Assume, contrary to what we claim, that $sum(T_\ell) < 0$. Let $T'_\ell$ be the maximal bad legal submatrix with respect to $R$ that contains $T_\ell$, and let $T''_\ell$ be the good legal submatrix that contains $T'_\ell$ and has twice its height and twice its width. Then $T''_\ell$ intersects either $B$ or $B'$, and in this intersection, the (legal) subcolumns of $T''_\ell$ strictly contain the subcolumns of $B$ or $B'$ (as in the case considered in the previous paragraph). But this contradicts the

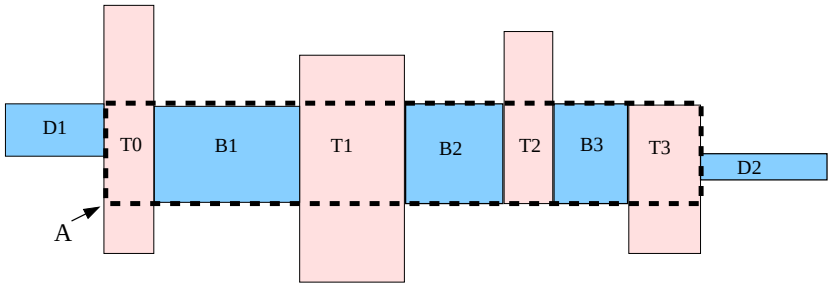


FIG. 7. *An illustration for Corollary* 8. *Here A covers the blocks $B_1$, $B_2$, and $B_3$ and borders the blocks $D_1$ and $D_2$. The submatrices $T_0$–$T_4$ are parts of larger blocks (that extend above and/or below A).*

maximality of $B$ or $B'$. □

By Lemma 7, we get the following corollary whose proof is illustrated in Figure 7.

COROLLARY 8. *Let A be a submatrix of R that covers a given block B. If on each of its sides A either borders a block with height smaller than the height of B or its border coincides with the border of R, then* $sum(A) \geq sum(B)$.

*Proof.* Let $B_1, \ldots, B_k$ be the set of maximal blocks that are covered by $A$ (where $B = B_i$ for some $1 \leq i \leq k$). Note that by definition of maximal blocks and covers, they are all of the same height, which is the height of $A$. Let $D_1$ and $D_2$ be two shorter blocks that border $A$ on the left side and the right side of $A$, respectively. (If there is no such block on one of the sides, then we think of the corresponding $D_i$ as having height 0.) Let $T_0, \ldots, T_k$ be the submatrices between these blocks (that have the same height as the blocks). That is, $T_0$ is between $D_1$ and $B_1$, $T_k$ is between $B_k$ and $D_2$, and for $1 \leq i \leq k-1$, $T_i$ is between $B_i$ and $B_{i+1}$. Then, by Lemma 7 and the fact that every block has a nonnegative sum we get that

$$sum(A) = \sum_{i=1}^{k} sum(B_i) + \sum_{i=0}^{k} sum(T_i) \geq sum(B). \qquad (6)$$ □

**5.4.2. Proving that Procedure 3 does not get stuck.** Recall that for each $1 \leq p \leq m$, $R_p$ is what remains of $R$ at the start of the $p$th iteration of Procedure 3. In particular, $R_1 = R$. In this section we show that the procedure does not "get stuck." That is, for each iteration $p$, Procedure 2 can be applied to the block $B_p$ selected in this iteration, and it is possible to update the designated sums of the rows that have been shortened by the removal of $B_p$. Note that since the blocks are selected according to increasing (nondecreasing) height, then in each iteration there indeed exists a unique cover $A_p$ of $B_p$ that is a maximal row-cover with respect to $R_p$.

For every $1 \leq p \leq m$, let $s_p$ be the minimum height of the maximal blocks of $R_p$, and let $s_0 = 1$. Observe that whenever $s_p$ increases, it does so by a factor of $2^k$ for some $k$. This is true because the columns of maximal blocks are legal subcolumns.

LEMMA 9. *For every $1 \leq p \leq m$, Procedure 2 can be applied to the block $B_p$ selected in $R_p$, and the updating process of the designated sum of rows can be applied. Moreover, if A is a submatrix of $R_p$ with height of at least $s_{p-1}$ whose columns are legal subcolumns and whose rows are maximal rows with respect to $R_p$, then* $\sum_{row\ L \in A} \overline{sum}(L) = sum(A)$.

*Proof.* Let $B_p$ be the block selected in iteration $p$, where $B_p$ is an $s \times t$ submatrix, and let $A_p$ be the maximal row-cover of $B_p$ with respect to $R_p$. As noted in

subsection 3.1, all that is required for Procedure 2 to work is the following:

(1) For every column $K$ in $B_p$, $sum(K) \geq 0$.
(2) For every row $L$ in $A_p$, $\overline{sum}(L) \geq 0$.
(3) $\sum_{\text{row } L \in A_p} \overline{sum}(L) \geq \sum_{\text{column } K \in B_p} sum(K)$.

In order for the updating process to succeed in step 3 of Procedure 3, we must have the following:

(4) For each $1 \leq \ell \leq s$, let $x_\ell$ be the sum of elements filled in the $\ell$th subrow of $B_p$, and let $L_\ell$ be the subrow of $A_p$ that covers this subrow of $B_p$. Then $\overline{sum}(L_\ell) - x_\ell \geq 0$.
(5) If $A_p \setminus B_p$ consists of the two submatrices $A'$ and $A''$ (between which resided $B$), then $sum(A') \geq 0$, $sum(A'') \geq 0$, and

$$\sum_{\text{row } L_\ell \in A_p} (\overline{sum}(L_\ell) - x_\ell) = sum(A') + sum(A'').$$

By Lemma 6, item (1) holds at the start of every iteration. In order to prove the other items for every $p$, we first extend and generalize item (2):

(2') Let $A$ be any submatrix in $R_p$ having height at least $s_{p-1}$ whose columns are legal subcolumns and whose rows are maximal rows with respect to $R_p$. Then for every row $L$ of $A$ we have $\overline{sum}(L) \geq 0$, and $\sum_{\text{row } L \in A} \overline{sum}(L) = sum(A)$.

Observe that if item (2') holds at the start of iteration $p$, then in particular it holds for $A_p$. Hence by Corollary 8

$$\sum_{\text{row } L \in A_p} \overline{sum}(L) = sum(A_p) \geq sum(B_p) \tag{7}$$

and so item (3) holds as well.

Furthermore, if items (1)–(3) hold at the start of iteration $p$, then Procedure 2 can be applied successfully. Thus item (4) necessarily holds by definition of Procedure 2. The first part of item (5), concerning the nonnegativity of $A'$ and $A''$, follows from Lemma 7 very similarly to the way Corollary 8 follows from this lemma. The second part of item (5) follows from item (2') holding for $A_p$ and the fact that $\sum_{\ell=1}^{s} x_\ell = sum(B_p)$ (since Procedure 2 completed successfully). Hence,

$$\sum_{\text{row } L_\ell \in A_p} (\overline{sum}(L_\ell) - x_\ell) = sum(A_p) - sum(B_p) = sum(A') + sum(A'') \tag{8}$$

as required.

Hence, it remains to prove that item (2') holds at the start of every iteration $p$. We do so by induction on $p$. Consider the base case, $p = 1$, so that $R_p = R_1 = R$. By the initialization of Procedure 3, for every maximal subrow $L$ of $R$, $\overline{sum}(L) = sum(L)$. By Lemma 6 (applied to the maximal legal subrows that partition $L$), we know that $\overline{sum}(L) \geq 0$. Furthermore, for every submatrix $A$ of $R$ having height of at least $s_{p-1} = s_0 = 1$ and whose rows are maximal subrows of $R$,

$$\sum_{\text{row } L \in A} \overline{sum}(L) = \sum_{\text{row } L \in A} sum(L) = sum(A) \tag{9}$$

as required.

Assuming that the induction claim holds for $p - 1$, we prove it for $p$. Consider any submatrix $A$ having height at least $s_{p-1}$ whose columns are legal subcolumns and

whose rows are maximal subrows with respect to $R_p$. If $A$ also consisted of maximal subrows with respect to $R_{p-1}$, then we are done by the induction hypothesis.

Otherwise, the block $B_{p-1}$ of height $s_{p-1}$ that was removed from $R_{p-1}$ bordered $A$ on one of its sides. Let $A^1, \ldots, A^q$ be the disjoint submatrices of height $s_{p-1}$ such that $A = \cup_{h=1}^{q} A^h$. That is, $A^1, \ldots, A^q$ are located one on top of the other (for an illustration, see Figure 8). In this case, all but at most one of these submatrices, say $A^q$, consisted of maximal subrows with respect to $R_{p-1}$, and $B_{p-1}$ bordered $A^q$.

For each of the submatrices $A^1, \ldots, A^{q-1}$ we can apply the induction hypothesis (item (2')). We get the following for each such $A^h$: (a) for every row $L$ in $A^h$, $\overline{sum}(L) \geq 0$; and (b) $\sum_{\text{row } L \in A^h} \overline{sum}(L) = sum(A^h)$.

As for $A^q$, assume without loss of generality that $B_{p-1}$ bordered $A^q$ from the right of $A^q$. Let $A'$ be the submatrix that bordered $B_{p-1}$ from the right of $B_{p-1}$ ($A'$ may be empty). This means that $A_{p-1}$ is of the form $A_{p-1} = A^q \cup B_{p-1} \cup A'$ (see Figure 8). But then, by definition of the updating rule and since it succeeded by the induction hypothesis (items (4) and (5)), we have that for every row $L$ in $A^q$, $\overline{sum}(L) \geq 0$ and $\sum_{\text{row } L \in A^q} \overline{sum}(L) = sum(A^q)$.

It follows that for every row $L$ in $A$ we have $\overline{sum}(L) \geq 0$ and

$$\sum_{\text{row } L \in A} \overline{sum}(L) = \sum_{h=1}^{q} \sum_{\text{row} L \in A^h} \overline{sum}(L) = \sum_{h=1}^{q} sum(A^h) = sum(A) \,. \tag{10}$$

The induction step is proven. □

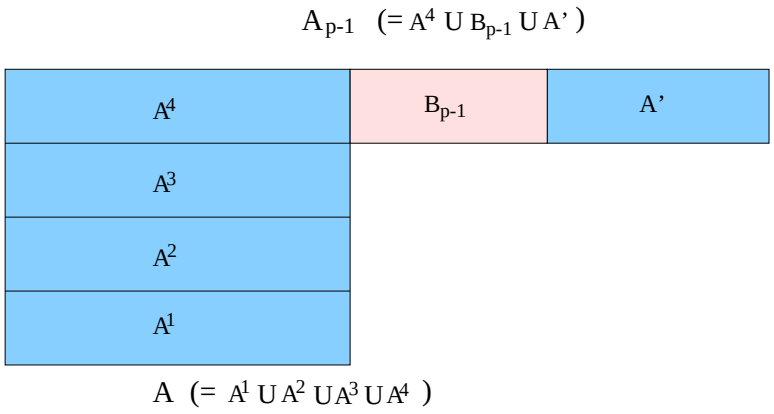


FIG. 8. *An illustration for the induction step in the proof of Lemma* 9 *(where* $q = 4$*).*

**5.4.3. Proving that Property 1 holds at the end of Procedure 3.** Finally, we have to show that when Procedure 3 terminates and $R$ is refilled with nonnegative values, then Property 1 holds. This will complete the proof of Theorem 3.

Let $\tilde{C}_V = \{\tilde{c}_{i,j}\}$ be the matrix resulting from the application of Procedure 3 to the matrix $C_V = \{c_{i,j}\}$. For any submatrix $T$ of $C_V$ (and in particular of $R$), we let $\widetilde{sum}(T)$ denote the sum of elements of $T$ in $\tilde{C}_V$. By definition of the procedure, $\widetilde{sum}(K) = sum(K)$ for every maximal legal subcolumn $K$ of $R$. Hence this holds also for every maximal subcolumn of $R$. We next prove a related claim concerning rows.

LEMMA 10. *For every subrow* $L$ *in* $R$ *such that* $L$ *is assigned* $\overline{sum}(L)$ *as a designated sum at some iteration of Procedure* 3*, we have that* $\widetilde{sum}(L) = \overline{sum}(L)$.

Observe that by combining Lemma 10 with Lemma 6 we get that for every maximal subrow $L$ of $R$, $\widetilde{sum}(L) = \overline{sum}(L) = sum(L)$.

*Proof.* Let $\mathcal{L}$ be the set of subrows $L$ of $R$ such that $L$ is assigned $\overline{sum}(L)$ as a designated sum at some iteration of Procedure 3. Observe that the set $\mathcal{L}$ consists exactly of those rows that are maximal subrows for some $R_p$. We prove the lemma by induction on the length of $L \in \mathcal{L}$. For the base of the induction, consider any subrow $L \in \mathcal{L}$ that is shortest among all subrows in $\mathcal{L}$. Since $L$ is shortest, it must be completely filled in a single iteration as part of a block $B$ (or otherwise there would be a shorter $L' \subset L$ with a designated sum $\overline{sum}(L')$). But by definition of the procedure, we get that $\widetilde{sum}(L) = \overline{sum}(L)$ as required.

Assume that the claim holds for every $L$ of length less than $\ell$; we prove it for $L$ having length $\ell$. Consider the first iteration after which $L$ became a maximal subrow (and thus received the designated sum $\overline{sum}(L)$) in which part of $L$ is filled. If all of $L$ is filled, then the induction claim follows as in the base case. Otherwise, let $x$ be the sum of elements that was filled in the part $P \subset L$. Let $L'$ and $L''$ be what remains of $L$ to the left and right of $P$, respectively. Then the procedure sets $\overline{sum}(L') + \overline{sum}(L'') = \overline{sum}(L) - x$. But $L'$ and $L''$ are strictly shorter than $L$, and therefore by the induction hypothesis $\widetilde{sum}(L') = \overline{sum}(L')$ and $\widetilde{sum}(L'') = \overline{sum}(L'')$. Thus $\widetilde{sum}(L) = \widetilde{sum}(L') + \widetilde{sum}(L'') + x = \overline{sum}(L') + \overline{sum}(L'') + x = \overline{sum}(L)$ as required. ∎

Definition 16 (boundary). *We say that a point $(i,j)$ is on the* boundary *of $R$ if $(i,j) \in R$, but either $(i+1,j) \notin R$, or $(i,j+1) \notin R$, or $(i+1,j+1) \notin R$. We denote the set of boundary points by $\mathcal{B}$.*

Definition 17. *For a point $(i,j)$, $1 \le i,j \le n$, let $R^{\le}(i,j)$ denote the subset of points $(i',j') \in R$, $i' \le i, j' \le j$, and let $sum^R(i,j) = \sum_{(i',j') \in R^{\le}(i,j)} c_{i',j'}$ and $\widetilde{sum}^R(i,j) = \sum_{(i',j') \in R^{\le}(i,j)} \tilde{c}_{i',j'}$.*

Property 1 and therefore Theorem 3 will follow directly from the next two lemmas.

Lemma 11. *For every point $(i,j) \in \mathcal{B}$, $\widetilde{sum}^R(i,j) = sum^R(i,j)$.*

*Proof.* Consider any point $(i,j) \in \mathcal{B}$, and let $U = R^{\le}(i,j)$. Let $\mathcal{C}(U) = \{B_1, \ldots, B_q\}$ be the minimal set of (maximal) blocks whose union contains $U$. For each $B_h \in \mathcal{C}(U)$ we know that $\widetilde{sum}(B_h) = sum(B_h)$. In particular, this is true for every $B_h \subset U$. Let $\mathcal{C}_1(U) = \{B_h \in \mathcal{C}(U) : B_h \subset U\}$. Hence we have that

$$\sum_{B_h \in \mathcal{C}_1(U)} \widetilde{sum}(B_h \cap U) = \sum_{B_h \in \mathcal{C}_1(U)} \widetilde{sum}(B_h) = \sum_{B_h \in \mathcal{C}_1(U)} sum(B_h)\,. \tag{11}$$

If every $B_h \in \mathcal{C}(U)$ is fully contained in $U$, then $\mathcal{C}_1(U) = \mathcal{C}(U)$ and we are done.

Otherwise, consider the remaining $B_h$'s in $\mathcal{C}(U) \setminus \mathcal{C}_1(U)$ (i.e., blocks that are not fully contained in $U$ but rather intersect it). Each of them contains either a column that is a subcolumn of column $j+1$ or a row that is a subrow of row $i+1$ (recall that $U = R^{\le}(i,j)$). Let the former subset be denoted $\mathcal{C}_2(U)$ and the latter $\mathcal{C}_3(U)$. Thus $\mathcal{C}_2(U)$ contains blocks that "intersect $U$ from the right," and $\mathcal{C}_3(U)$ contain blocks that "intersect $U$ from the top." See, for example, Figure 9.

It is important to note that $\mathcal{C}_2(U) \cap \mathcal{C}_3(U) = \emptyset$: If there existed a block $B_h \in \mathcal{C}_2(U) \cap \mathcal{C}_2(U)$, it would necessarily contain both $(i,j)$ and the three neighboring points, $(i+1,j)$, $(i,j+1)$, and $(i+1,j+1)$. But this contradicts the fact that $(i,j)$ is a boundary point.

For each $B_h \in \mathcal{C}_2(U)$, $B_h \cap U$ is a subset of maximal legal subcolumns with respect to $R$ (since each $B_h \in \mathcal{C}_2(U)$ cannot extend beyond row $i$). Let $\mathcal{K}_2(U)$ denote the set

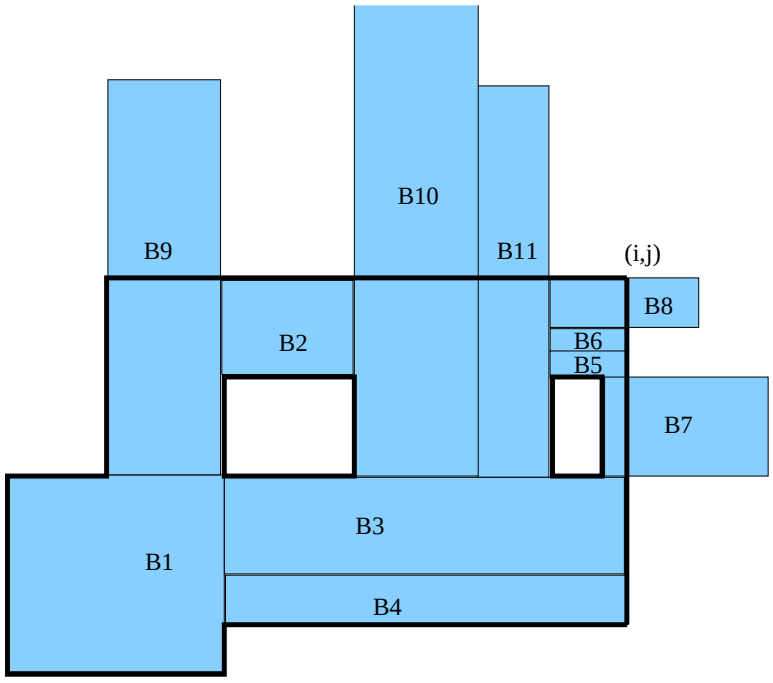


FIG. 9. *An illustration for the proof of Lemma* 11. *The solid line denotes the outline of* $U = R^{\leq}(i,j)$, *where point* $(i,j)$ *is in the top-right corner. Blocks* $B_1$–$B_6$ *are fully contained in* $U$ *and therefore belong to* $\mathcal{C}_1(U)$. *Blocks* $B_7$ *and* $B_8$ *belong to* $\mathcal{C}_2(U)$ *and blocks* $B_9$–$B_{11}$ *belong to* $\mathcal{C}_3(U)$. *Block* $B_{10}$ *is twice the height of* $B_9$ *and* $B_{11}$ *and thus "extends out of the figure."*

of all maximal legal subcolumns that belong to $\bigcup_{B_h \in \mathcal{C}_2(U)}(B_h \cap U)$. Since for every maximal legal subcolumn $K$ it holds that $\widetilde{sum}(K) = sum(K)$, we have that

$$\sum_{B_h \in \mathcal{C}_2(U)} \widetilde{sum}(B_h \cap U) = \sum_{K \in \mathcal{K}_2(U)} \widetilde{sum}(K) = \sum_{K \in \mathcal{K}_2(U)} sum(K). \tag{12}$$

Next consider the blocks $B_h \in \mathcal{C}_3(U)$. Let $\mathcal{L}_3(U)$ be the set of subrows in $U$ that are maximal subrows with respect to $\bigcup_{B_h \in \mathcal{C}_3(U)}(B_h \cap U)$. Thus, $\bigcup_{B_h \in \mathcal{C}_3(U)}(B_h \cap U) = \bigcup_{L \in \mathcal{L}_3} L$. We next observe that for every $B_h \in \mathcal{C}_3(U)$, all blocks that border $B_h$ and belong either to $\mathcal{C}_1(U)$ or to $\mathcal{C}_2(U)$ must be strictly shorter than $B_h$. This follows from the definition of legal subcolumns. Hence, the blocks in $\mathcal{C}_1(U)$ and $\mathcal{C}_2(U)$ are all removed before the blocks in $\mathcal{C}_3(U)$.

For each subrow in $\mathcal{L}_3(U)$ there exists the first iteration $p$ in which it becomes a maximal subrow with respect to $R_p$ (following the removal of some block in $\mathcal{C}_1(U) \cup \mathcal{C}_2(U)$ from $R_{p-1}$). We partition the rows in $\mathcal{L}_3(U)$ accordingly. Let $\mathcal{L}_3^p(U)$ denote all subrows in $\mathcal{L}_3(U)$ that are maximal subrows with respect to $R_p$ but were not maximal subrows with respect to $R_{p-1}$. Observe that, in particular, $\mathcal{L}_3^1(U)$ is the set of subrows in $\mathcal{L}_3(U)$ that were already maximal subrows with respect to $R$. By this definition the subrows in $\mathcal{L}_3^p(U)$ constitute a submatrix of height $s_{p-1}$. By the second part of Lemma 9, $\sum_{L \in \mathcal{L}_3^p(U)} \widetilde{sum}(L) = \sum_{L \in \mathcal{L}_3^p(U)} sum(L)$, and by applying Lemma 10 we

get that $\sum_{L\in\mathcal{L}_3(U)} \widetilde{sum}(L) = \sum_{L\in\mathcal{L}_3(U)} sum(L)$. Therefore,

$$\sum_{B_h\in\mathcal{C}_3(U)} \widetilde{sum}(B_h\cap U) = \sum_{L\in\mathcal{L}_3(U)} \widetilde{sum}(L) = \sum_{L\in\mathcal{L}_3(U)} sum(L). \tag{13}$$

By combining (11)–(13) we get

$$\begin{aligned}\widetilde{sum}(U) &= \sum_{B_h\in\mathcal{C}(U)} \widetilde{sum}(B_h\cap U)\\ &= \sum_{q=1}^{3}\sum_{B_h\in\mathcal{C}_q(U)} \widetilde{sum}(B_h\cap U)\\ &= \sum_{B_h\in\mathcal{C}_1(U)} sum(B_h) + \sum_{K\in\mathcal{K}_2(U)} sum(K) + \sum_{L\in\mathcal{L}_3(U)} sum(L)\\ &= sum(U) \qquad \square\end{aligned}$$

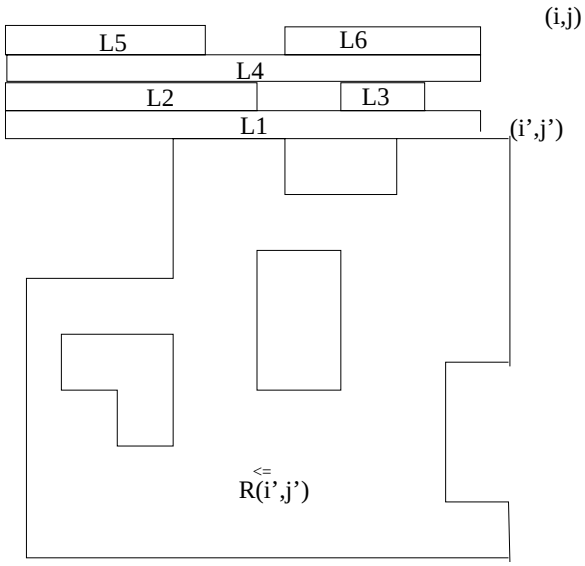


FIG. 10. *An illustration for the proof of Lemma* 12. *The point* $(i', j')$ *is as defined in the proof, and the rows* $L_1, \ldots, L_6$ *are all maximal subrows of* $R$ *that belong to rows* $i'+1, \ldots, i$ *and end by column* $j$ *(that is, the set* $\mathcal{L}(i, i', j)$*).*

LEMMA 12. *Let* $(i, j)$ *be any point such that* $(i, j) \notin R$. *Then* $\widetilde{sum}^R(i, j) = sum^R(i, j)$.

*Proof.* Let $(i', j') \in R$, $i' < i$, $j' \le j$, be the point for which $j'$ is maximized, and if there are several such points, let it be the one amongst them for which $i'$ is maximized. Thus, $(i', j')$ is maximal in the sense that for every $(i'', j'')$, $i'' < i$, $j'' \le j$ such that $(i'', j'') > (i', j')$ it holds that $(i'', j'') \notin R$. Furthermore, among all such maximal points it is the right-most one (i.e., it belongs to the column with the highest index). By definition, $(i', j')$ belongs to $\mathcal{B}$, since $(i'+1, j')$ necessarily does not belong to $R$. Let $\mathcal{L}(i, i', j)$ be the subset of all maximal subrows of $R$ that belong to rows $i'+1, \ldots, i$ and end by column $j$. Then $\widetilde{sum}^R(i, j) = \widetilde{sum}^R(i', j') + \sum_{L\in\mathcal{L}(i,i',j)} \widetilde{sum}(L)$. By

applying Lemma 11 and Lemma 10, we get that $\widetilde{sum}^R(i,j) = sum^R(i,j)$. For an illustration, see Figure 10. □

**5.5. Distribution matrices.** As noted in the introduction, a subfamily of inverse Monge matrices that is of particular interest is the class of *distribution matrices*. A matrix $V = \{v_{i,j}\}$ is said to be a distribution matrix if there exists a nonnegative *density matrix* $D = \{d_{i,j}\}$ such that every entry $v_{i,j}$ in $V$ is of the form $v_{i,j} = \sum_{k \le i} \sum_{\ell \le j} d_{k,\ell}$. In particular, if $V$ is a distribution matrix, then the corresponding density matrix $D$ is simply the matrix $C'_V$ (as defined in section 3). Hence, in order to test that $V$ is a distribution matrix, we simply run our algorithm for inverse Monge matrix on $C'_V$ instead of $C_V$.

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