A pricing problem under Monge property

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Abstract

We study a pricing problem where buyers with non-uniform demand purchase one of many items. Each buyer has a known benefit for each item and purchases the item that gives the largest utility, which is defined to be the difference between the benefit and the price of the item. The optimization problem is to decide on the prices that maximize total revenue of the seller. This problem is also called the optimal product line design problem in the absence of competition.

Even though the general problem is known to be NP-hard, it can be solved efficiently under some natural assumptions on customer benefits. In this paper we study properties of optimal solutions and present a dynamic programming algorithm when customer benefits satisfy the Monge property. The same algorithm can also be used to solve the problem under the additional requirement that all buyers should be served.

1 Introduction

In this paper, we study a pricing problem where given a collection of buyers and a collection of available items, the seller wants to decide on the prices of the items so as to maximize his revenue. The buyers are assumed to have individual demand levels and the items are available in unlimited supply. Each buyer has a given benefit for each item and purchases the item that maximizes his utility provided that the utility, which is defined as the difference between the benefit and the price of the item, is non-negative. This is a relatively common utility model in economic theory known as the linear utility model, see[9]. We also consider a variation of this problem where each buyer has to be served and therefore prices have to be set in such a way that for each buyer there is at least one item that gives a non-negative utility.

A possible application of this model could be in the telecommunications industry where a mobile phone operator has to decide on which calling plans to offer and how to price these plans to maximize revenue. A "buyer", in this context, corresponds to a group of customers with similar needs and purchasing power. This is a reasonable model as marketing departments of these companies typically segregate their customer base into segments with similar behavior. The estimated number of customers in each group gives the demand of each "buyer". Similarly, "items" correspond to different calling plans of which each buyer can choose at most one. Clearly, our model in this context overlooks the potential effects of other service providers and assumes that the customer base is indifferent to external events. This problem is also known as the *optimal product line design* problem in the absence of competition. The term "product line" here refers to a collection of substitutable products. See [11] for an early reference and [10] for recent references.

Formally, we define the *pricing problem* as follows: Let $I = \{1, \ldots, n\}$ denote the set of buyers, $J = \{1, \ldots, m\}$ denote the set of items, $D_i \in R_+$ denote the demand of buyer $i \in I$, and $B_{i,j} \in R_+$ denote the benefit of buyer $i \in I$ for obtaining item $j \in J$. Without loss of generality, we assume that problem data satisfies $B \neq 0$ and $D > 0$. Given a price vector $P \in R_+^m$, the *assignment vector* $\alpha \in J^n$, maps each buyer to one of the items that maximizes his utility. In other words, for $i \in I$,

$$B_{i,\alpha(i)} - P_{\alpha(i)} \geq B_{i,j} - P_j$$

for all $j \in J$. If there is a tie, we assume that the buyer is mapped to the item with the higher benefit, and therefore with higher price. We break further ties by mapping the buyer to the item with the smaller index. Buyer i purchases item $\alpha(i)$ provided that $B_{i,\alpha(i)} \geq P_{\alpha(i)}$. If $B_{i,\alpha(i)} < P_{\alpha(i)}$, we assume that the buyer does not purchase any of the items and his demand is unsatisfied. The resulting total revenue for the seller is

$$\sum_{i \in I : B_{i,\alpha(i)} \geq P_{\alpha(i)}} D_i P_{\alpha(i)}.$$

Therefore, the optimization problem becomes,

$$z = \max \Big\{ \sum_{i \in I : B_{i,\alpha(i)} \geq P_{\alpha(i)}} D_i P_{\alpha(i)} \ : \ P \in R_+^m, \alpha \in J^n, B_{i,\alpha(i)} - P_{\alpha(i)} \geq B_{i,j} - P_j \ \forall i \in I,\ j \in J \Big\}.$$

The pricing problem [2] and its variations [7, 8]are known to be NP-hard.

It is also possible to formulate this problem as a bi-linear mixed-integer program (see [4] for a similar formulation):

$$\text{Maximize} \quad \sum_{i \in I} D_i \sum_{j \in J} x_{i,j}\, p_j$$

$$\begin{array}{lrcll}
\text{Subject to:} & \sum_{j \in J} x_{i,j} & \leq & 1 & \text{for all } i \in I \\
& u_i & \geq & B_{i,j} - p_j & \text{for all } i \in I,\ j \in J \\
& u_i & = & \sum_{j \in J} x_{i,j}\,(B_{i,j} - p_j) & \text{for all } i \in I \\
& u,\, p & \geq & 0, & \\
& x & \in & \{0,1\}^{n \times m}. &
\end{array}$$

where $x_{i,j}$ denotes whether or not buyer i buys item j; p_j denotes the price of item j; and, u_i denotes the utility of buyer i. As $u \geq 0$, if all items give negative utility for a buyer i, the solution has $x_{i,j} = 0$ for all $j \in J$.

We note that it is also possible to produce a linear formulation for the pricing problem by introducing (*i*) a new set of variables $y_{ij} \geq 0$ that stand for the product $x_{ij}p_j$, and, (*ii*) a new set of linearization constraints based on $\max\{0, p_j - M_j(1 - x_{ij})\} \leq y_{ij} \leq \min\{p_j, M_j x_{ij}\}$, where M_j is a valid bound on p_j such as $\max_{i \in I}\{B_{i,j}\}$.

The paper is organized as follows: In the next section we describe the Monge property and related assumptions on buyer benefits. We then study structure of optimal solutions under these assumptions. In Section 3, we study the pricing problem when all buyers have to served and present a dynamic programming algorithm. Finally, in Section 4, we relax the restriction that all buyers have to be served and show that the pricing problem still admits a dynamic programming algorithm if buyer benefits satisfy the extended Monge property.

2 Monge property and basic results

In the rest of the paper, we study the pricing problem when the benefit matrix B is a *Monge matrix.* A matrix $C \in R^{n \times m}$ is said to satisfy the *Monge property* if

$$C_{ij} + C_{rs} \geq C_{is} + C_{rj} \text{ for all } 1 \leq i < r \leq m,\ 1 \leq j < s \leq n. \tag{1}$$

To be precise, this property is actually called the *inverse* Monge property, which is same as the Monge property after reordering the rows and columns of the matrix C. It is easy to show that inequality (1) holds provided that Monge property holds for adjacent rows and columns

$$C_{ij} + C_{i+1,j+1} \geq C_{i,j+1} + C_{i+1,j} \text{ for all } 1 \leq i < m,\ 1 \leq j < n. \tag{2}$$

An early reference for an optimization application of this property dates back to 1781 when the French mathematician Gaspard Monge studied a basic transportation problem. This property was later formalized and attributed to Monge by Alan Hoffman [6] in 1961. In this paper, Hoffman also shows that the well-known transportation problem can be solved using a greedy algorithm, called the *northwest rule*, if the cost matrix satisfies the Monge property. Since then a number of optimization problems have been studied in the presence of the this property, including some generalizations of the transportation problem [3]. We refer the reader to [1] for a survey.

We say that a matrix $C \in R^{n \times m}$ satisfies the *extended Monge property* provided that all of the following three assumptions hold:

Assumption 1. For all $1 \leq i < n$ and $1 \leq j < m$

$$C_{i,j} - C_{i,j+1} \geq C_{i+1,j} - C_{i+1,j+1}.$$

Assumption 2. For all $1 \leq i \leq n$ and $1 \leq j < m$

$$C_{i,j} \geq C_{i,j+1}.$$

Assumption 3. For all $1 \leq i < n$ and $1 \leq j \leq m$

$$C_{i,j} \geq C_{i+1,j}.$$

When applied to the benefit matrix B, Assumptions 2 and 3 order items and buyers in such a way that items with smaller indices are more beneficial for all buyers; and buyers with smaller indices obtain larger benefit from obtaining any item and therefore, they obtain larger utility in the optimal solution independent of how the prices are set. Combined with the Monge property of Assumption 1, these assumptions imply that buyers with smaller indices value money less, that is, they are relatively indifferent to prices, and therefore, they are more likely to buy items that they value most.

Finally, we also note that if Assumption 1 holds for the benefit matrix B, then, (i) $B_{i,j} - B_{i,j+1} \geq B_{n,j} - B_{n,j+1}$ and therefore Assumption 2 reduces to requiring $B_{n,j} - B_{n,j+1} \geq 0$ for all $j \in J \setminus \{m\}$, and, (ii) $B_{i,j} - B_{i+1,j} \geq B_{i,m} - B_{i+1,m}$ and Assumption 3 reduces to $B_{i,m} - B_{i+1,m} \geq 0$ for all $i \in I \setminus \{n\}$.

2.1 Basic properties independent of the assumptions

We next describe a natural property of the assignment vectors associated with similar price vectors.

Proposition 1 *Let $P^1, P^2 \in R^m$ be two price vectors and let $\alpha^1, \alpha^2 \in J^n$ be the corresponding assignment vectors. Furthermore, let $P^1_j = P^2_j$ for all $j \in J \backslash \{k\}$, and $P^2_k > P^1_k$. Then $\alpha^1(i) = \alpha^2(i)$ for all $i \in I$ such that $\alpha^1(i) \neq k$.*

Proof. Let $i \in I$ be such that $\alpha^1(i) = t \neq k$. As $B_{i,t} - P^1_t \geq B_{i,j} - P^1_j$ for all $j \in J$ by definition and $P^2 \geq P^1$, we have $B_{i,t} - P^2_t = B_{i,t} - P^1_t \geq B_{i,j} - P^2_j$ for all $j \in J$ as well. ■

In other words, if the price of an item is increased, the only change in the assignment vector is due to the buyers abandoning this item. Next, we describe a basic property of optimal price vectors. Given an optimal assignment vector, the following observation relates optimal prices to buyer benefits. This observation has stronger implications under Assumptions 1 and 2.

Proposition 2 *Let $P \in R^m$ be a price vector and $\alpha \in J^n$ be the corresponding assignment vector. Let $\bar{I} = \{i \in I : B_{i,\alpha(i)} \geq P_{\alpha(i)}\}$, $\bar{J} = \{j \in J : j = \alpha(i)$ for some $i \in \bar{I}\}$, and let $P' = \min_{j \in \bar{J}}\{P_j\}$ be the price of the cheapest item that has a buyer. Let $\bar{I}' = \{i \in \bar{I} : P_{\alpha(i)} = P'\}$, then either*

$$\min_{i \in \bar{I}'}\{B_{i,\alpha(i)}\} \;=\; P',$$

or, the total revenue $\sum_{i \in \bar{I}} D_i P_{\alpha(i)}$ can be improved while serving the same set of customers $\bar{I}$.

Proof. Assume $\delta = \min_{i \in \bar{I}'}\{B_{i,\alpha(i)}\} - P' > 0$. In this case prices for all $j \in \bar{J}' = \{j \in \bar{J} : P_j = P'\}$ can be increased uniformly by $\delta/2$. Notice that with the new prices $B_{i,\alpha(i)} \geq P'_{\alpha(i)}$ still holds for all $i \in \bar{I}$. The assignment vector α does not necessarily reflect buyer preferences correctly for the new prices, but by Proposition 1 none of the buyers in $\bar{I} \setminus \bar{I}'$ change their assignment after seeing the new prices, and all buyers in $\bar{I}'$ now either pay more for the same item or buy more expensive items. Therefore, total revenue increases. ■

2.2 Structural properties under one of the assumptions

We next present a strong structural property of the assignment vector when the buyer befits satisfy the Monge property. Note that the following observation holds for any price vector, including sub-optimal ones.

Proposition 3 *Under Assumption 1, for any given price vector $P \in R^m$, the corresponding assignment vector $\alpha \in J^n$ has no crossing. In other words, $\alpha(i+1) \geq \alpha(i)$ for all $i \in I \setminus \{n\}$.*

Proof. Remember that, by definition, for $i \in I$, $\alpha(i)$ satisfies $B_{i,\alpha(i)} - P_{\alpha(i)} \geq B_{i,j} - P_j$ for all $j \in J$, and in case of a tie, the buyer is assigned to the item with the higher benefit (price), and if there is still a tie, to the item with the smallest index.

Assume that α does not satisfy the claim for some $i \in I \setminus \{n\}$ and let $\alpha(i+1) = j$ and $\alpha(i) = k$ where $j < k$. If both i and $i+1$ are indifferent to items j and k, that is, if $B_{i,k} - P_k = B_{i,j} - P_j$, and $B_{i+1,k} - P_k = B_{i+1,j} - P_j$, then, as the ties are broken the same way, $\alpha(i) = \alpha(i+1)$, a contradiction. We can therefore assume that at least one of i or $i+1$ strictly prefers the assigned item.

In this case, if buyer i has a strict preference, i.e. $B_{i,k} - P_k > B_{i,j} - P_j$ we combine this inequality with with $B_{i+1,j} - P_j \geq B_{i+1,k} - P_k$, to obtain

$$B_{i,k} - B_{i,j} > P_k - P_j \geq B_{i+1,k} - B_{i+1,j}.$$

More precisely, $B_{i+1,j} - B_{i+1,k} > B_{i,j} - B_{i,k}$ where $k > j$. Note this inequality can also be established if buyer $i+1$ has a strict preference for j. This inequality can be re-written as

$$\sum_{l=j}^{k-1} B_{i+1,l} - B_{i+1,l+1} > \sum_{l=j}^{k-1} B_{i,l} - B_{i,l+1}$$

which is inconsistent with Assumption 1 when considered term by term. ■

A direct consequence of the above result is that under Assumption 1, there is an optimal assignment vector that partitions the buyers into contiguous groups with the same assigned item. In other words, for any $i \in I \setminus \{n\}$, either $\alpha(i) = \alpha(i+1)$, or, $\alpha(i) < \alpha(i+1)$. Therefore, there is a partition of $I = \cup_{l=1}^{k} C_l$ where $C_l = \{s_l, \ldots, t_l\}$ and $\alpha(i) = \alpha_l$ for all $i \in C_l$. Furthermore, $\alpha_l \leq \alpha_{l+1}$ for $l = 1, \ldots, k-1$. This observation is one of the main building blocks of the dynamic programming algorithm described later. We demonstrate this structure in Figure 1 for $k = 4$.

We next present a basic property of prices of items that have at least one potential buyer in a solution. Note that the following observation is also valid for any price vector, not only the optimal one.

Proposition 4 *Given a price vector $P \in R^m$ and the corresponding assignment vector $\alpha \in J^n$, let $\bar{J} = \{j \in J : \ j = \alpha(i) \ \text{ for some } i \in I\}$ denote the set of desired items. Then, under Assumption 2, if $j_1 < j_2$ for $j_1, j_2 \in \bar{J}$, then*

$$P_{j_1} > P_{j_2}.$$

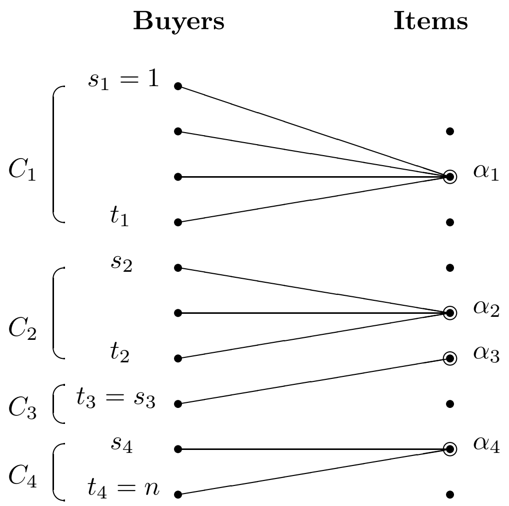

Figure 1: Partitioning of the buyers due to Monge property

Proof. Assume the claim is wrong, and $P_{j_1} \leq P_{j_2}$. Note that there can be no $i \in I$ for which $\alpha(i) = j_2$ as $B_{i,j_1} \geq B_{i,j_2}$ and $P_{j_2} \geq P_{j_1}$ imply that $B_{i,j_1} - P_{j_1} \geq B_{i,j_2} - P_{j_2}$. This gives a contradiction even when $B_{i,j_1} - P_{j_1} = B_{i,j_2} - P_{j_2}$, as ties are broken in the favor of the item with the smaller index. ∎

Note that there is a high level of degeneracy with regards to the optimal prices as items without a buyer in the solution, that is $j \notin \bar{J}$, can assume arbitrarily high prices without changing the assignment vector or total revenue. It is possible to extend Proposition 4 to show a complete ordering of item prices, including the ones without a buyer. In other words, we can show that there exists an optimal price vector P' with the same assignment vector and total revenue that satisfies $P'_j \geq P'_{j+1}$ for all $j \in J \setminus \{m\}$. This can simply be achieved by setting the prices of the items in $J \setminus \bar{J}$ as high as possible without violating the ordering condition.

Next, we present a basic property of assignment vectors under Assumption 3.

Proposition 5 *Given a price vector $P \in R^m$ and the corresponding assignment vector $\alpha \in J^n$, let $\bar{I} = \{i \in I \,:\, B_{i,\alpha(i)} \geq P_{\alpha(i)}\}$. Then, under Assumption 3, $\bar{I} = \{1, 2, , \ldots, i'\}$ for some $i' \in I$.*

Proof. Let $i \in \bar{I} \setminus \{1\}$, and consider buyer $i - 1 \in I$. As $B_{i,\alpha(i)} - P_{\alpha(i)} \geq 0$ and $B_{i-1,\alpha(i)} \geq B_{i,\alpha(i)}$ by Assumption 3, we have $B_{i-1,\alpha(i)} - P_{\alpha(i)} \geq 0$ and therefore $i - 1 \in \bar{I}$ as well. Combining this this observation with the fact that $\bar{I} \neq \emptyset$, the proof is complete. ∎

3 A DP algorithm for a restricted pricing problem

In this section, we study a restriction of the pricing problem where the price vector P is required to satisfy the following property

$$\max_{j\in J}\{B_{i,j} - P_j\} \geq 0$$

for all $i \in I$. In other words, all buyers need to be assigned to an item with non-negative utility. In this case total revenue simply becomes $\sum_{i\in I} D_i P_{\alpha(i)}$.

Throughout this section, we also assume that buyer benefits $B \in R^{n\times m}$ satisfy the extended Monge property.

We denote the optimal price vector by $P^* \in R^m$ and the corresponding assignment vector by $\alpha^* \in J^n$. Without loss of generality, we also assume that the optimal assignment vector partitions buyers into contiguous groups as described by Proposition 3. In other words, $I = \cup_{l=1}^k C_l$ where $C_l = \{s_l, \ldots, t_l\}$ and $\alpha^*(i) = \alpha_l$ for all $i \in C_l$. Furthermore, $\alpha_l \leq \alpha_{l+1}$ for $l = 1, \ldots, k-1$.

Finally, we will use the following proposition to assume that $\alpha_1 = 1$.

Proposition 6 *Without loss of generality,* $\alpha^*(1) = 1$.

Proof. Let $\alpha^*(1) = \alpha_1 \neq 1$. By Proposition 4, α_1 is the most expensive item that has a buyer, that is, $P^*_{\alpha_1} \geq P^*_{\alpha_l}$ for $l = 1, \ldots, k$. Furthermore, by Assumption 2, $B_{1,1} \geq B_{1,\alpha_1}$ and as $\alpha_1 \neq 1$, we also have $P^*_{\alpha_1} B_{1,\alpha_1} > B_{1,1} - P^*_1$. Since ties are broken in favor of the item with the smaller index, $P^*_1 > P^*_{\alpha_1}$.

Let P^{**} be the price vector obtained from P^{**} by reducing the price of item 1 to $P^*_{\alpha_1}$ and keeping the remaining prices the same. Add let α^{**} be the corresponding assignment vector. Clearly, $\alpha^{**}(1) = 1$. In addition, by Proposition 1, if $\alpha^{**}(i) \neq \alpha^*(i)$, for some $i \in I$, then $\alpha^{**}(i) = 1$. In other words, the only assignments that would change are for buyers who would switch to item 1. As the new price of item 1 is equal to the highest price in the original solution, the total revenue can only increase.

Therefore, P^{**} is also an optimal price vector and $\alpha^{**}(i) = 1$. ■

We next present two observation that relate optimal prices to the assignment vector using buyer benefits.

Proposition 7 *All customers are served if and only if* $P^*_{\alpha^*(n)} = B_{n,\alpha^*(n)}$.

Proof. Let $\alpha^*(n) = \alpha_k$. As Assumption 3 holds, if $B_{n,\alpha_k} = P^*_{\alpha_k}$, then $B_{i,\alpha_k} \geq P^*_{\alpha_k}$ for all $i \in I$, and therefore, all buyers are served.

Next, assume that all buyers are served. Note that, by definition, α_k is the item with the largest index that has a buyer. As P^* is optimal, by Proposition 4, $P^*_{\alpha_k}$ is the price of the cheapest item that has a buyer and by Proposition 2, $P^*_{\alpha_k} = \min_{i\in C_k}\{B_{i,\alpha(i)}\} = B_{n,\alpha_k}$. ■

Proposition 8 *For any $l \in \{1, 2, \ldots, k-1\}$, buyer t_l is indifferent between items α_l and α_{l+1}. In other words, if $k > 1$, then $B_{t_l,\alpha_l} - P^*_{\alpha_l} = B_{t_l,\alpha_{l+1}} - P^*_{\alpha_{l+1}}$.*

Proof. Assume the claim does not hold for some $l \in \{1, 2, \ldots, k-1\}$. Since $\alpha^*_{t_l} = \alpha_l$, we have $B_{t_l,\alpha_l} - P_{\alpha_l} \geq B_{t_l,\alpha_{l+1}} - P_{\alpha_{l+1}}$. If the claim is wrong, t_l must strictly prefer item α_l, i.e. $P^*_{\alpha_l} < B_{t_l,\alpha_l} + P^*_{\alpha_{l+1}} - B_{t_l,\alpha_{l+1}}$.

In this case, if $P^*_{\alpha_l}$ is increased to $P^*_{\alpha_{l+1}} + B_{t_l,\alpha_l} - B_{t_l,\alpha_{l+1}}$ customer t_l and therefore, by Proposition 1 and Proposition 3, no other customer would abandon item α_l to buy a cheaper item. Note that $P^*_{\alpha_l}$, and therefore the total revenue can be strictly increased, a contradiction. ∎

A straight forward observation based on Proposition 8 is the following.

Corollary 9 *For any $l \in \{1, 2, \ldots, k-1\}$, $P^*_{\alpha_l} - P^*_{\alpha_{l+1}} = B_{t_l,\alpha_l} - B_{t_l,\alpha_{l+1}}$.*

Combining this observation with the fact that $B_{n,\alpha_k} = P^*_{\alpha_k}$ by Proposition 7, it is possible to express the optimal prices using buyer benefits provided that the optimal assignment vector is known. That is,

$$P^*_{\alpha_l} = P^*_{\alpha_k} + \sum_{q=l}^{k-1} \left(P^*_{\alpha_q} - P^*_{\alpha_{q+1}} \right) = B_{n,\alpha_k} + \sum_{q=l}^{k-1} \left(B_{l_q,\alpha_q} - B_{l_q,\alpha_{q+1}} \right).$$

for all $l \in \{1, 2, \ldots, k-1\}$.

Furthermore, total revenue can also be expressed using buyer benefits. For ease of notation, define $D(a, b) = \sum_{i=a}^{b} D_i$ and let $\alpha_{l+1} = m + 1$ and $B_{n,m+1} = 0$ and define $P^*_{m+1} = 0$. We can now write

$$\begin{aligned}
\sum_{i=1}^{n} D_i P^*(\alpha^*(i)) &= \sum_{h=1}^{k} D(s_h, t_h) P^*(\alpha_h) \\
&= \sum_{h=1}^{k} D(s_h, t_h) \left(\sum_{l=h}^{k} \left(P^*(\alpha_l) - P^*(\alpha_{l+1}) \right) \right) \\
&= \sum_{l=1}^{k} D(1, t_l) \left(P^*(\alpha_l) - P^*(\alpha_{l+1}) \right) \\
&= \sum_{l=1}^{k} D(1, t_l) \left(B_{t_l,\alpha_l} - B_{t_l,\alpha_{l+1}} \right)
\end{aligned}$$

provided that $k > 1$. If $k = 1$, then $\alpha_1 = 1$ and $z^* = D(1, n) B_{n,1}$.

In other words, given an optimal assignment vector, it is possible to compute the resulting objective function value (total revenue) without explicitly constructing the optimal price vector. This observation forms the basis of our dynamic programming recursion as the problem now reduces to searching for an optimal assignment vector which is a combinatorial object. We next define a family of problems obtained by perturbing the original problem.

Let $\mathcal{PP}(a,b)$ denote the pricing problem with the buyer set reduced to $\bar{I} = \{a, \dots, n\}$, and the item set reduced to $\bar{J} = \{b, \dots, m\}$. Furthermore, the demands $\bar{D}$ for the reduced problem are obtained by setting $\bar{D}_a = D(1,a)$ and $\bar{D}_i = D_i$ for $i > a+1$. The requirement that all buyers should be served still holds for $\mathcal{PP}(a,b)$. Notice that if the original befit matrix B satisfies any of the Assumptions 1-3, the benefit matrix of the reduced problem also satisfies the same assumptions.

Let $z^*(a,b)$ denote the optimal value of $\mathcal{PP}(a,b)$. Using this notation, the original problem can now be called $\mathcal{PP}(1,1)$ and the optimal value of the original problem z^* is equal to $z^*(1,1)$. In the following two Propositions, we relate the optimal value of the reduced problem to the optimal value of the original problem.

Proposition 10 *Let $b \in J$, then,*

$$z^* \geq D_1 \left(B_{1,1} - B_{1,b}\right) + z^*(2,b).$$

Proof. Given the optimal solution to the reduced problem $\mathcal{PP}(2,b)$, it is possible to obtain a solution to the original problem as follows: By Proposition 6, we can assume that the optimal assignment vector of $\mathcal{PP}(2,b)$ assigns buyer 2 to item b. If $b > 1$, the price vector of the reduced problem is augmented by setting P_1 to $P_b + B_{1,1} - B_{1,b} \geq P_b \geq 0$ and setting prices of all remaining items $\{2, \dots, b-1\}$ (if any) to a large number to obtain a price vector for the original problem. Next, we augment the optimal assignment vector of the reduced problem by assigning buyer 1 to item 1 to construct an assignment vector to the original problem. Note that the augmented price vector is consistent with the augmented assignment vector. Also note that all buyers are still served as $B_{1,1} - P_1 = B_{1,b} - P_b \geq B_{2,b} - P_b \geq 0$.

The objective value of this solution is $z^*(2,b) + D_1 \left(B_{1,1} - B_{1,b}\right)$. ■

Proposition 11 *Let α^* be the optimal assignment vector for the pricing problem. Then,*

$$z^* = D_1 \left(B_{1,1} - B_{1,\alpha^*(2)}\right) + z^*(2, \alpha^*(2)).$$

Proof. First note that if $\alpha^*(2) = 1$, that is, $\alpha^*(1) = \alpha^*(2)$, the claim is correct as $z^* = z^*(2,1)$. On the other hand, assume $\alpha^*(2) \neq 1$ and therefore the group of buyers assigned to item 1 only consists of buyer 1, i.e., $C_1 = \{1\}$.

In this case, it is possible to truncate the solution to the pricing problem by deleting buyer 1 and items $1, \dots, \alpha^*(2) - 1$ to obtain a solution to $\mathcal{PP}(2, \alpha^*(2))$, implying

$$z^*(2, \alpha^*(2)) \geq z^* - D_1 \left(P_1^* - P_{\alpha^*(2)}^*\right) = z^* - D_1 \left(B_{1,1} - B_{1,\alpha^*(2)}\right)$$

Observing that $z^*(2, \alpha^*(2)) \leq z^* - D_1 \left(B_{1,1} - B_{1,\alpha^*(2)}\right)$ by Proposition 10 completes the proof. ■

Combining Proposition 10 and 11, we have therefore shown that:

$$z^* = z^*(1,1) = \max_{b \in J} \left\{ D_1 \left(B_{1,1} - B_{1,b}\right) + z^*(2,b) \right\} \tag{3}$$

which leads to the following dynamic programming recursion:

Proposition 12 *For all $i \in I \setminus \{1,n\}$ and $j \in J$*

$$z^*(i,j) = \max_{b \in \{j,\ldots,m\}} \Big\{ D(1,i)\,(B_{i,j} - B_{i,b}) + z^*(i+1,b) \Big\}$$

Proof. It suffices to apply inequality (3) after renumbering buyers and items in $\mathcal{PP}(i,j)$ starting from 1 instead of i and j. ∎

Finally, as $\mathcal{PP}(n,j)$ is a trivial problem with a single buyer, we have $z^*(n,j) = D(1,n)B_{n,j}$ by Proposition 6. Therefore under Monge property and with the additional requirement that all buyers are served, the pricing problem can be solved by the dynamic programming algorithm below.

Algorithm 1

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for $j = m$ **down to** 1 **do**

$z^*(n,j) = D(1,n)B_{n,j}$

for $i = n-1$ **down to** 2 **do**

for $j = m$ **down to** 1 **do**

$$z^*(i,j) = \max_{b \in \{j,\ldots,m\}} \Big\{ D(1,i)\,(B_{i,j} - B_{i,b}) + z^*(i+1,b) \Big\}$$

end do

end do

$$z^* = z^*(1,1) = \max_{b \in J} \Big\{ D_1 \big(B_{(1,1} - B_{1,b} \big) + z^*(2,b) \Big\}$$

After applying Algorithm 1, it is straight forward to construct the optimal solution as follows: First set $\alpha_1^* = 1$. Then, for the remaining buyers $i \in I \setminus \{1\}$, sequentially set α_i^* to the item $b \geq \alpha_{i-1}^*$ that satisfies

$$z^*(i,b) = z^*(i-1,\alpha_{i-1}^*) - D(1,i-1)\big(B_{i-1,\alpha_{i-1}^*} - B_{i-1,b}\big).$$

The optimal prices can now be constructed by Proposition 2 and Corollary 9. Therefore, we have shown the following result.

Theorem 13 *If all buyers have to be served and buyer benefits satisfy Assumptions 1,2 and 3, then the pricing problem can be solved by Algorithm 1 in $O(nm^2)$ time using a state space of size $O(nm)$.*

4 A DP algorithm under the extended Monge property

In this section we show that Algorithm 1, with minor modifications, solves the pricing problem if buyer benefits satisfy extended Monge property. In other words, we drop the requirement that all buyers need to be served. In this case, by Proposition 5 , the collection of buyers with non-negative utility in the optimal solution are $\bar{I} = \{i \in I \,:\, B_{i,\alpha^*(i)} \geq P^*_{\alpha^*(i)}\} = \{1, 2, , \ldots, i'\}$ for some $i' \in I$. Therefore, it is possible to solve the pricing problem under the extended Monge assumption by solving n (one for each possible value of i') instances of the restricted problem studied in Section 3. Each restricted instance has a subset of the original customers, namely $\{1, \ldots, i'\}$, all of which have to be served. Maximum revenue obtained by the solutions of these restricted problems gives the maximum revenue for the original problem. This algorithm can actually be improved as described below.

Notice that if a matrix $C \in R_+^{n\times m}$ satisfies the extended Monge property then $C' \in R_+^{n\times m+1}$ also satisfies the extended Monge property where C' is obtained from C by adding a column of zeroes at the end. Therefore, by simply adding an artificial item which has zero benefit for every buyer, it is possible to generate a new instance of the pricing problem that satisfies Assumptions 1, 2 and 3. Furthermore, without loss of generality, the price vector can now be required to serve all buyers as the price of the new item can always be set to 0 without changing the value of the optimal solution. It is easy to obtain a one-to-one correspondence between the solutions of the original problem instance and the new instance by setting the price of the new item to zero and assigning all unserved buyers by the original problem to the new item. Clearly corresponding solutions of the two problems have the same objective function value. Therefore, we have the following result.

Theorem 14 *If buyer benefits satisfy the extended Monge property, the pricing problem can be solved by dynamic programming in $O(nm^2)$ time using a state space of size $O(nm)$.*

Proof. Given an instance of the pricing problem, create a new instance with the same set of buyers and the item set $J' = J \cup \{m+1\}$. The benefit matrix $B' \in R^{n\times m+1}$ is obtained from $B \in R^{n\times m}$ by setting $B_{i,m+1} = 0$ for the missing entries. Applying Algorithm 1 to the new instance gives the optimal solution to the original problem. Buyers $i \in I$ with $\alpha^*(i) = m+1$ are not served. ∎

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