Lower-Stretch Spanning Trees

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Introduction

- Graph Embedding on Tree Metrics

  Average $O(\log^2 n \log \log n)$ stretch.

  \[
  \text{stretch}_T(u, v) = \frac{\text{dist}_T(u,v)}{d(u,v)}
  \]

  \[
  \text{ave-stretch}_T(E) = \frac{1}{|E|} \sum_{(u,v) \in E} \text{stretch}_T(u, v)
  \]

- Star Decomposition
Notation

- The **boundary** of $S$, $\partial S$: the set of edges with exactly one endpoint in $S$.
- The **volume** of a set of edges $F$, $\text{vol}(F)$: the size of the set $F$.
- The **volume** of a set of vertices $S$, $\text{vol}(S)$: the number of edges incident to $S$.
- The **ball shell** around a vertex $v$, $\text{BS}(r, v)$: the set of vertices "right" outside $B(r, v)$.
- The **cost** (weight) of an edge, the length is $d(e) = 1/w(e)$. 
Low-Cost Star-Decomposition

- A multiway partition \( \{V_0, V_1, \ldots, V_k\} \) with center \( x_0 \in V_0 \) is a \textit{star-decomposition}:
  - subgraphs induced by \( V_i \) are connected.
  - \( x_i \in V_i \) is connected to a vertex \( y_i \in V_0 \) by an edge \( (x_i, y_i) \in E \). (bridge).

![Diagram of a star-decomposition with a center \( x_0 \) and \( k \) connected subgraphs, each containing a vertex \( y_i \) connected to \( x_0 \) by an edge.](image)
Low-Cost Star-Decomposition

- Let $r = rad_G(x_0)$, and $r_i = rad_{V_i}(x_i)$. For $\delta, \epsilon \leq 1/2$, a star-decomposition is a $(\delta, \epsilon)$-star-decomposition if
  - $\delta r \leq r_0 \leq (1 - \delta)r$
  - $r_0 + d(x_i, y_i) + r_i \leq (1 + \epsilon)r$

- The *cost* of the star-decomposition is

$$\text{cost}(\partial(V_0, V_1, \ldots, V_k))$$

the sum of cost of the edges between the sets.
Low-Cost Star-Decomposition

Let $G = (V, E, w)$ be a connected weighted graph and $x_0 \in V$. For every positive $\epsilon \leq 1/2$,

$$(\{V_0, V_1, \ldots, V_k\}, x, y) = \text{starDecomp}(G, x_0, 1/3, \epsilon),$$

in time $O(m + n \log n)$, returns a $(1/3, \epsilon)$-star-decomposition of $G$ with center $x_0$ of cost

$$\text{cost}(\partial(V_0, V_1, \ldots, V_k)) \leq \frac{6m \log_2(m + 1)}{\epsilon \cdot \text{rad}_G(x_0)}$$

- $\delta r \leq r_0 \leq (1 - \delta)r$
- $r_0 + d(x_i, y_i) + r_i \leq (1 + \epsilon)r$
$O(\log^3 m)$ Average Stretch Tree

- Algorithm for Unweighted Graphs
  
  Fix $\alpha = (2 \log_{4/3}(\hat{n} + 6))^{-1}$.
  
  $T = \text{UnweightedLowStretchTree}(G, x_0)$

1. If $|V| \leq 2$, return $G$.
2. Set $\rho = \text{rad}_G(x_0)$
3. $\langle \{V_0, V_1, \ldots, V_k\}, x, y \rangle = \text{StarDecomp}(G, x_0, 1/3, \alpha)$
4. For each $i$,
   set $T_i = \text{UnweightedLowStretchTree}(G(V_i), x_i)$.
5. Set $T = \cup_i T_i \cup_i (y_i, x_i)$. 
$O(\log^3 m)$ Average Stretch Tree

- **Analysis**

  Depth of recursion: $O(\log_{4/3} n)$

  \[
  \text{rad}_{R_t(G)}(x_0) \leq (1 + \alpha)^t \text{rad}_G(x_0) \leq \sqrt{e} \cdot \text{rad}_G(x_0).
  \]

  \[
  \sum_{(u,v)\in \partial(V_0,...,V_k)} \text{stretch}_T(u,v) \leq \sum_{(u,v)\in \partial(V_0,...,V_k)} (\text{dist}_T(x_0,u) + \text{dist}_T(x_0,v))
  \leq \sum_{(u,v)\in \partial(V_0,...,V_k)} 2\sqrt{e} \cdot \text{rad}_G(x_0)
  \leq 2\sqrt{e} \cdot \text{rad}_G(x_0) \frac{6m \log_2(\hat{m} + 1)}{\alpha \cdot \text{rad}_G(x_0)}
  \]

  \[
  \sum_{(u,v)\in E} \text{stretch}_T(u,v) = O(\hat{m} \log^3 \hat{m}).
  \]
Algorithm for Weighted Graphs

Fix $\beta = (2 \log_{4/3}(\hat{n} + 32))^{-1}$. $T = \text{LowStretchTree}(G, x_0)$

1. If $|V| \leq 2$, return $G$.
2. Set $\rho = \text{rad}_G(x_0)$
3. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the graph by contracting all edges in $G$ with length less than $\beta \rho / \hat{n}$.
4. $(\{\tilde{V}_0, \ldots, \tilde{V}_k\}, x, y) = \text{StarDecomp}(\tilde{G}, x_0, 1/3, \beta)$
5. For each $i$, let $V_i$ be the preimage of $\tilde{V}_i$, and $(x_i, y_i)$ be one of the preimage of $(\tilde{x}_i, \tilde{y}_i)$.
6. For each $i$, set $T_i = \text{LowStretchTree}(G(V_i), x_i)$.
7. Set $T = \bigcup_i T_i \cup_i (y_i, x_i)$. 

$O(\log^3 m)$ Average Stretch Tree

Fix $\beta = (2 \log_{4/3}(\hat{n} + 32))^{-1}$. $T = \text{LowStretchTree}(G, x_0)$
Average Stretch Tree

Analysis

Let $t = 2 \log_{4/3}(\hat{n} + 32)$ and $\rho_t = \text{rad}_{R_t(G)}(x_0)$

$$\rho_t \leq \sqrt{e} \cdot \text{rad}_G(x_0)$$

Each component has radius at most $\rho(3/4)^t \leq \rho/n^2$.

Each edge appears at most $\log_{4/3}((2\hat{n}/\beta) + 1)$ recursion depths.

The total contribution to the stretch at level $t$ is

$$O(\text{vol}(E_t) \log^2 \hat{m})$$
Star Decomposition

- Concentric System

A family of vertex sets \( L = \{ L_r \subseteq V : r \in R^+ \cup \{0\} \} \).

1. \( L_0 \neq \emptyset \),

2. \( L_r \subseteq L_{r'} \) for all \( r \leq r' \),

3. if a vertex \( u \in L_r \) and \((u, v) \in E\), then \( v \in L_{r+d(u,v)} \).
Star Decomposition

- Concentric System

A family of vertex sets $\mathcal{L} = \{ L_r \subseteq V : r \in \mathbb{R}^+ \cup \{0\} \}$.

1. $L_0 \neq \emptyset$,
2. $L_r \subseteq L_{r'}$ for all $r \leq r'$,
3. if a vertex $u \in L_r$ and $(u, v) \in E$, then $v \in L_{r + d(u, v)}$.

- Property: For every two reals $0 \leq \lambda \leq \lambda'$, there exists a real $r \in [\lambda, \lambda')$ such that

$$
\text{cost}(\partial(L_r)) \leq \frac{\text{vol}(L_r)}{\lambda' - \lambda} \max \left[ 1, \log_2 \left( \frac{m}{\text{vol}(E(L_r))} \right) \right]
$$
Star Decomposition

- Proof of the property:

\[ \text{cost}(\partial(L_r)) \leq \frac{\text{vol}(L_r)}{\lambda' - \lambda} \max \left[ 1, \log_2 \left( \frac{m}{\text{vol}(E(L_r))} \right) \right] \]

Sort the vertices according to the distances to the center.

Let \( \mu_i = \text{vol}(E(B_i)) + \sum (v_j,v_k) \in E : j \leq i < k \frac{r_i-r_j}{r_k-r_j} \).

\[ \mu_{i+1} = \mu_i + \text{cost}(\partial(B_i))(r_{i+1} - r_i) \]

Let \( r_{a-1} \leq \lambda < \lambda' \leq r_{b+1} \), and \( \eta = \log_2 \left( \frac{m}{\text{vol}(E(B_{a-1}))} \right) \)

Prove there exists \( i \in [a - 1, b] \) such that

\[ \text{cost}(\partial(B_i)) \leq \mu_i \eta / (\lambda' - \lambda). \]
Star Decomposition

- \( r = \text{BallCut}(G, x_0, \rho, \delta) \)

1. Set \( r = \delta \rho \)

2. While \( \text{cost}(\partial(B(r, x_0))) > \frac{\text{vol}(B(r, x_0)) + 1}{(1 - 2\delta)\rho} \log_2(m + 1) \),
   Find the next vertex \( v \) and set \( r = \text{dist}(x_0, v) \).

- Result:

\[ \frac{\rho}{3} \leq r \leq 2\frac{\rho}{3} \]

\[ \text{cost}(\partial(V_0)) > \frac{3(\text{vol}(V_0) + 1)\log_2(|E| + 1)}{\rho} \]
Star Decomposition

- Ideals and Cones
  
  For set $S \subseteq V$,
  
  $$F(S) = \{(u \to v) : (u, v) \in E, \text{dist}(u, S) + d(u, v) = \text{dist}(v, S)\}$$

- The **ideal** of $v$, $I_S(v)$, induced by $S$, is the set of vertices that reachable from $v$ in $F(S)$

- The **cone** of width $l$ around $v$ induced by $S$, $C_S(l, v)$, is the set of vertices in $V$ that can be reached from $v$ by a path, the sum of lengths of whose edges not in $F(S)$ is at most $l$. 
Star Decomposition

- Cones are concentric
  \[ r = \text{ConeCut}(G, v, \lambda, \lambda', S) \]
  1. Set \( r = \lambda \) if \( \text{vol}(E(C_S(\lambda, v))) = 0 \),
     Set \( \mu = (\text{vol}(C_S(r, v)) + 1) \log_2(m + 1) \).
     otherwise,
     Set \( \mu = \text{vol}(C_S(r, v)) \log_2(m/\text{vol}(E(C_S(\lambda, v)))) \).
  2. While \( \text{cost}(\partial(C_S(r, v))) > \mu/(\lambda' - \lambda) \),
     Find the next vertex \( w \) minimize \( \text{dist}(w, C_S(r, v)) \) and set
     \( r = r + \text{dist}(w, C_S(r, v)) \).

\[ r \in [\lambda, \lambda') \]

\[ \text{cost}(\partial(C_S(r, v))) \leq \frac{\text{vol}(C_S(r, v))}{\lambda' - \lambda} \max \left[ 1, \log_2 \frac{m}{\text{vol}(E(C_S(r, v)))} \right] \]
Star Decomposition

- **Final Algorithm**

\[
(V_0, \ldots, V_k, \mathbf{x}, \mathbf{y}) = \text{StarDecomp}(G, x_0, \delta, \epsilon)
\]

1. Set \( \rho = \text{rad}_G(x_0) \); \( r_0 = \text{BallCut}(G, x_0, \rho, \delta) \) and \( V_0 = B(r_0, x_0) \).
2. Let \( S = BS(r_0, x_0) \).
3. Set \( G' = (V', E', w') = G(V - V_0) \).
4. Set \( (V_1, \ldots, V_k, \mathbf{x}) = \text{ConeDecomp}(G', S, \epsilon \rho/2) \).
5. For each \( i \in [1 : k] \), set \( y_k \) to be a vertex in \( V_0 \) such that \( (x_k, v_k) \in E \) and \( y_k \) is on a shortest path from \( x_0 \) to \( x_k \).

\[
(V_1, \ldots, V_k, \mathbf{x}) = \text{ConeDecomp}(G, S, \Delta)
\]

1. Set \( G_0 = G, S_0 = S, k = 0 \).
2. While \( S_k \) is not empty
   (a) \( k = k + 1; x_k \in S_k; r_k = \text{ConeCut}(G_{k-1}, x_k, 0, \Delta, S_{k-1}) \).
   (b) Set \( V_k = C_{S_{k-1}}(r_k, x_k) \); \( G_k = G(V - \bigcup_{i=1}^{k} V_k) \), \( S_k = S_{k-1} - V_k \).
3. Set \( \mathbf{x} = (x_1, \ldots, x_k) \).
Star Decomposition

- Cost

\[
\begin{align*}
\text{cost}(\partial(V_0)) & > \frac{3(\text{vol}(V_0) + 1) \log_2(|E| + 1)}{\rho} \\
\text{cost}(E(V_j, V - \bigcup_{i=0}^{j} V_i)) & \leq \frac{2(1 + \text{vol}(V_j)) \log_2(m + 1)}{\epsilon \rho} \\
\text{cost}(\partial(V_0, \ldots, V_K)) & \leq \sum_{j=0}^{k} \text{cost}(E(V_j, V - \bigcup_{i=0}^{j} V_i)) \\
& \leq \frac{2 \log_2(m + 1)}{\epsilon \rho} \sum_{j=0}^{k} (\text{vol}(V_j) + 1) \\
& \leq \frac{6m \log_2(m + 1)}{\epsilon \rho}
\end{align*}
\]