Lower-Stretch Spanning Trees

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COMP670P

Introduction

- Graph Embedding on Tree Metrics

  Average $O(\log^2 n \log\log n)$ stretch.

  $\text{stretch}_T(u, v) = \frac{\text{dist}_T(u,v)}{d(u,v)}$

  ave-stretch$_T(E) = \frac{1}{|E|} \sum_{(u,v)\in E} \text{stretch}_T(u, v)$

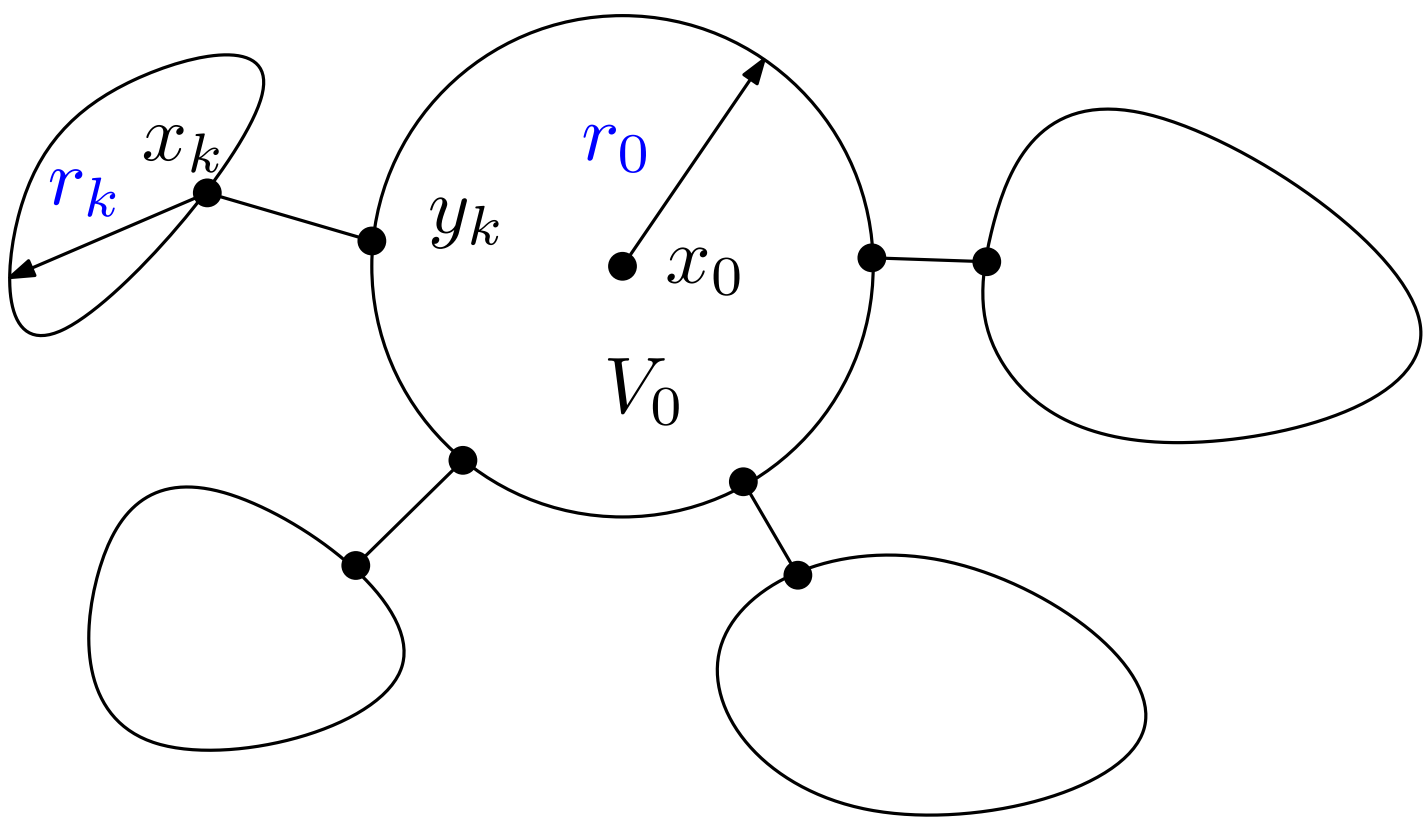
- Star Decomposition

Notation

- The *boundary* of S, ∂S: the set of edges with exactly one endpoint in S.
- The *volumn* of a set of edges F, $\mathrm{vol}(F)$: the size of the set F.
- The *volumn* of a set of vertices S, $\mathrm{vol}(S)$: the number of edges incident to S.
- The *ball shell* around a vertex v, $\mathrm{BS}(r, v)$: the set of vertices "right" outside $B(r, v)$.
- The *cost* (weight) of an edge, the length is $d(e) = 1/w(e)$.

Low-Cost Star-Decomposition

- A multiway partition $\{V_0, V_1, \ldots, V_k\}$ with center $x_0 \in V_0$ is a *star-decomposition*:
  - subgraphs induced by V_i are connected.
  - $x_i \in V_i$ is connected to a vertex $y_i \in V_0$ by an edge $(x_i, y_i) \in E$. (*bridge*).



Low-Cost Star-Decomposition

- Let $r = rad_G(x_0)$, and $r_i = rad_{V_i}(x_i)$. For $\delta, \epsilon \leq 1/2$, a star-decomposition is a (δ, ϵ)-star-decomposition if
  - $\delta r \leq r_0 \leq (1-\delta)r$
  - $r_0 + d(x_i, y_i) + r_i \leq (1+\epsilon)r$
- The *cost* of the star-decomposition is

  $\mathrm{cost}(\partial(V_0, V_1, \ldots, V_k))$, the sum of cost of the edges between the sets.

Low-Cost Star-Decomposition

- Let $G = (V, E, w)$ be a connected weighted graph and $x_0 \in V$. For every positive $\epsilon \leq 1/2$,

$$(\{V_0, V_1, \ldots, V_k\}, \mathbf{x}, \mathbf{y}) = \mathrm{starDecomp}(\mathbf{G}, \mathbf{x_0}, \mathbf{1/3}, \epsilon),$$

  in time $O(m + n \log n)$, returns a $(1/3, \epsilon)$-star-decomposition of G with center x_0 of cost

$$\mathrm{cost}(\partial(V_0, V_1, \ldots, V_k)) \leq \frac{6m \log_2(m+1)}{\epsilon \cdot \mathrm{rad}_G(x_0)}$$

  - $\delta r \leq r_0 \leq (1-\delta) r$
  - $r_0 + d(x_i, y_i) + r_i \leq (1+\epsilon) r$

$O(\log^3 m)$ Average Stretch Tree

- Algorithm for Unweighted Graphs

  Fix $\alpha = (2\log_{4/3}(\hat{n}+6))^{-1}$.

  $T = \text{UnweightedLowStretchTree}(G, x_0)$

  1. If $|V| \leq 2$, return G.
  2. Set $\rho = rad_G(x_0)$
  3. $(\{V_0, V_1, \ldots, V_k\}, \mathbf{x}, \mathbf{y}) = \text{StarDecomp}(\mathbf{G}, \mathbf{x_0}, \mathbf{1/3}, \alpha)$
  4. For each i,
     set $T_i = \text{UnweightedLowStretchTree}(G(V_i), x_i)$.
  5. Set $T = \cup_i T_i \cup_i (y_i, x_i)$.

$O(\log^3 m)$ Average Stretch Tree

- Analysis

  Depth of recursion: $O(\log_{4/3} n)$

  $$\mathrm{rad}_{R_t(G)}(x_0) \le (1+\alpha)^t \mathrm{rad}_G(x_0) \le \sqrt{e} \cdot \mathrm{rad}_G(x_0).$$

$$\begin{aligned}
\sum_{(u,v)\in\partial(V_0\ldots,V_k)} \mathrm{stretch}_T(u,v) &\le \sum_{(u,v)\in\partial(V_0\ldots,V_k)} (\mathrm{dist}_T(x_0,u) + \mathrm{dist}_T(x_0,v)) \\
&\le \sum_{(u,v)\in\partial(V_0,\ldots,V_k)} 2\sqrt{e} \cdot \mathrm{rad}_G(x_0) \\
&\le 2\sqrt{e} \cdot \mathrm{rad}_G(x_0) \frac{6m \log_2(\hat{m}+1)}{\alpha \cdot \mathrm{rad}_G(x_0)}
\end{aligned}$$

$$\sum_{(u,v)\in E} stretch_T(u,v) = O(\hat{m} \log^3 \hat{m}).$$

$O(\log^3 m)$ Average Stretch Tree

- Algorithm for Weighted Graphs

  Fix $\beta = (2\log_{4/3}(\hat{n}+32))^{-1}$. $T = \mathrm{LowStretchTree}(G, x_0)$

  1. If $|V| \le 2$, return G.
  2. Set $\rho = rad_G(x_0)$
  3. Let $\tilde{G} = (\tilde{V}, \tilde{E})$ be the graph by contracting all edges in G with length less than $\beta\rho/\hat{n}$.
  4. $(\{\tilde{V}_0, \ldots, \tilde{V}_k\}, \mathbf{x}, \mathbf{y}) = \mathrm{StarDecomp}(\tilde{\mathbf{G}}, \mathbf{x_0}, \mathbf{1/3}, \beta)$
  5. For each i, let V_i be the preimage of $\tilde{V}_i$, and (x_i, y_i) be one of the preimage of $(\tilde{x}_i, \tilde{y}_I)$.
  6. For each i,
     set $T_i = \mathrm{LowStretchTree}(G(V_i), x_i)$.
  7. Set $T = \cup_i T_i \cup_i (y_i, x_i)$.

$O(\log^3 m)$ Average Stretch Tree

- Analysis

  Let $t = 2\log_{4/3}(\hat{n} + 32)$ and $\rho_t = \mathrm{rad}_{R_t(G)}(x_0)$

  $\rho_t \leq \sqrt{e} \cdot \mathrm{rad}_G(x_0)$

  Each component has radius at most $\rho(3/4)^t \leq \rho/n^2$..

  Each edge appears at most $\log_{4/3}((2\hat{n}/\beta) + 1)$ recursion depths.

  The total contribution to the stretch at level t is

$$O(vol(E_t) \log^2 \hat{m})$$

Star Decomposition

- Concentric System

  A family of vertex sets $\mathcal{L} = \{L_r \subseteq V \,:\, r \in R^+ \cup \{0\}\}$.

  1. $L_0 \neq \emptyset$,
  2. $L_r \subseteq L_{r'}$ for all $r \leq r'$,
  3. if a vertex $u \in L_r$ and $(u, v) \in E$, then $v \in L_{r+d(u,v)}$.

Star Decomposition

- Concentric System

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  3. if a vertex $u \in L_r$ and $(u, v) \in E$, then $v \in L_{r+d(u,v)}$.

- Property: For every two reals $0 \leq \lambda \leq \lambda'$, there exists a real $r \in [\lambda, \lambda')$ such that

$$cost(\partial(L_r)) \leq \frac{\text{vol}(L_r)}{\lambda' - \lambda} \max\left[1, \log_2\left(\frac{m}{\text{vol}(E(L_r))}\right)\right]$$

Star Decomposition

- Proof of the property:

$$cost(\partial(L_r)) \leq \frac{\mathrm{vol}(L_r)}{\lambda' - \lambda} \max\left[1, \log_2\left(\frac{m}{\mathrm{vol}(E(L_r))}\right)\right]$$

Sort the vertices according to the distances to the center.

Let $\mu_i = \mathrm{vol}(E(B_i)) + \sum_{(v_j,v_k)\in E: j\leq i<k} \frac{r_i - r_j}{r_k - r_j}$.

$$\mu_{i+1} = \mu_i + \mathrm{cost}(\partial(B_i))(r_{i+1} - r_i)$$

Let $r_{a-1} \leq \lambda < \lambda' \leq r_{b+1}$, and $\eta = \log_2\left(\frac{m}{\mathrm{vol}(E(B_{a-1}))}\right)$

Prove there exists $i \in [a-1, b]$ such that

$$\mathrm{cost}(\partial(B_i)) \leq \mu_i \eta / (\lambda' - \lambda).$$

Star Decomposition

- $r = BallCut(G, x_0, \rho, \delta)$

  1. Set $r = \delta\rho$
  2. While $\mathrm{cost}(\partial(B(r, x_0))) > \frac{\mathrm{vol}(B(r,x_0))+1}{(1-2\delta)\rho}\log_2(m+1)$,
     Find the next vertex v and set $r = dist(x_0, v)$.

- Result:

$$\rho/3 \leq r \leq 2\rho/3$$

$$\mathrm{cost}(\partial(V_0)) > \frac{3(\mathrm{vol}(V_0)+1)\log_2(|E|+1)}{\rho}$$

Star Decomposition

- Ideals and Cones

  For set $S \subseteq V$,

  $$F(S) = \{(u \to v) \,:\, (u,v) \in E, \mathrm{dist}(u,S) + d(u,v) = \mathrm{dist}(v,S)\}$$

- The *ideal* of v, $I_S(v)$, induced by S, is the set of vertices that reachable from v in $F(S)$

- The *cone* of width l around v induced by S, $C_S(l,v)$, is the set of vertices in V that can be reached from v by a path, the sum of lengths of whose edges not in $F(S)$ is at most l.

Star Decomposition

- Cones are concentric

  $r = ConeCut(G, v, \lambda, \lambda', S)$

  1. Set $r = \lambda$ if $\mathrm{vol}(E(C_S(\lambda, v))) = 0$,
     Set $\mu = (\mathrm{vol}(C_S(r, v)) + 1)\log_2(m + 1)$.
     otherwise,
     Set $\mu = \mathrm{vol}(C_S(r, v))\log_2(m/\mathrm{vol}(E(C_S(\lambda, v)))$.
  2. While $\mathrm{cost}(\partial(C_S(r, v))) > \mu/(\lambda' - \lambda)$,
     Find the next vertex w minimize $\mathrm{dist}(w, C_S(r, v))$ and set
     $r = r + dist(w, C_S(r, v))$.

  $r \in [\lambda, \lambda')$

$$\mathrm{cost}(\partial(C_S(r, v))) \leq \frac{\mathrm{vol}(C_S(r, v))}{\lambda' - \lambda} \max\left[1, \log_2 \frac{m}{\mathrm{vol}(E(C_S(r, v)))}\right]$$

Star Decomposition

- Final Algorithm

$(\{V_0, \ldots, V_k\}, \mathbf{x}, \mathbf{y}) \;= \text{StarDecomp}(G, x_0, \delta, \epsilon)$

1. Set $\rho = \text{rad}_G(x_0)$; $r_0 = \text{BallCut}(G, x_0, \rho, \delta)$ and $V_0 = B(r_0, x_0)$.
2. Let $S = BS(r_0, x_0)$;
3. Set $G' = (V', E', w') = G(V - V_0)$.
4. Set $(\{V_1, \ldots, V_k, \mathbf{x}\}) = \text{ConeDecomp}(G', S, \epsilon\rho/2)$;
5. For each $i \in [1 : k]$, set y_k to be a vertex in V_0 such that $(x_k, v_k) \in E$ and y_k is on a shortest path from x_0 to x_k

$(\{V_1, \ldots, V_k, \mathbf{x}\}) \;= \text{ConeDecomp}(G, S, \Delta)$

1. Set $G_0 = G, S_0 = S, k = 0$.
2. While S_k is not empty
   (a) $k = k + 1$; $x_k \in S_k$; $r_k = \text{ConeCut}(G_{k-1}, x_k, 0, \Delta, S_{k-1})$.
   (b) Set $V_k = C_{S_{k-1}}(r_k, x_k)$; $G_k = G(V - \cup_{i=1}^{k} V_k)$,$S_k = S_{k-1} - V_k$.
3. Set $\mathbf{x} = (x_1, \ldots, x_k)$.

Star Decomposition

- Cost

$$\mathrm{cost}(\partial(V_0)) > \frac{3(\mathrm{vol}(V_0)+1)\log_2(|E|+1)}{\rho}$$

$$\mathrm{cost}(E(V_j, V - \cup_{i=0}^{j} V_i)) \leq \frac{2(1+\mathrm{vol}(V_j))\log_2(m+1)}{\epsilon\rho}$$

$$\begin{aligned}
\mathrm{cost}(\partial(V_0,\ldots,V_K)) &\leq \sum_{j=0}^{k} \mathrm{cost}(E(V_j, V - \cup_{i]=}^{j} V_i)) \\
&\leq \frac{2\log_2(m+1)}{\epsilon\rho}\sum_{j=0}^{k}(\mathrm{vol}(V_j)+1) \\
&\leq \frac{6m\log_2(m+1)}{\epsilon\rho}
\end{aligned}$$