A Tight Bound on Approximating Arbitrary Metrics by Tree Metrics

Jittat Fakcheroenphol   Satish Rao   Kunal Talwar

STOC 2003, JCSS 2004

Presented by Jian XIA

for COMP670P: Topics in Theory: Metric Embeddings and Algorithms
Spring 2007, HKUST

May 8, 2007

Random Tree Embedding

Given a metric (V, d). Let S be a family of metrics over V, and let D be a distribution over S. We say that (S, D) α-*probabilistically approximates* a metric (V, d), if

- every metric in S dominates d;
  ($d'(u, v) \geq d(u, v)$, for every $u, v \in V$ and every metric $d' \in S$.)
- for every $u, v \in V$,

$$\mathbf{E}_{d' \in (S, D)}[d'(u, v)] \leq \alpha \cdot d(u, v).$$

We call α the distortion.

Question

What is the distortion for probabilistic approximation by dominating trees?

Known Results

- Embedding C_n (unit weight n-cycle) into a spanning tree requires distortion at least $n-1$.
- Embedding C_n into a tree requires $\Omega(n)$ distortion. [Rabinovich and Raz, 95]
- C_n can be embedded into a distribution of dominating trees with distortion $2(1-1/n)$. [Karp, 89]
- $2^{O(\sqrt{\log n \log\log n})}$ distortion for graph metrics, using spanning trees. [Alon *et al.*, 95]
- $O(\log^2 n)$ distortion; there exists a graph requiring $\Omega(\log n)$ distortion. [Bartal, 96]
    - Note: Tree metrics can be isometrically embedded into ℓ_1
- $O(\log n \log\log n)$ distortion [Bartal, 98]
- This paper closes the gap!
- $O(\log^2 n \log\log n)$ distortion for graph metrics, using spanning trees. [Elkin *et al.*, 05]

Hierarchical Cut Decomposition

- assumption: the smallest distance in the given n-point metric space (V, d) is strictly more than 1; and the diameter of the metric is $\Delta = 2^\delta$.
- A *hierarchical cut decomposition* of (V, d) is a sequence of $\delta + 1$ nested cut decompositions $D_0, D_1, \ldots, D_\delta$ such that
  - $D_\delta = \{V\}$,
  - D_i is a 2^i-cut decomposition, and a refinement of D_{i+1}.(that is, each set in D_{i+1} is a disjoint union of some sets of D_i.)

  where, given a parameter r, an r-cut decomposition of (V, d) is a partitioning of V into clusters, each centered around a vertex and having radius at most r.
- Property
  - the diameter of each cluster in D_i (referred as *level* i *cluster*) is at most 2^{i+1}
  - each cluster in D_0 is a singleton vertex.
  - a hierarchical cut decomposition naturally corresponds to a rooted tree.

Corresponding tree

- The vertices of the tree have the form (S, i), where $S \in D_i$, and $i = 0, 1, \ldots, \delta$.
- The root is (V, δ)
- The children of a vertex (S, i) are $(T, i-1)$ with $T \in D_{i-1}$ and $T \subseteq S$
- The edge connecting (S, i) to $(T, i-1)$ has length 2^i.

The tree metric d_T is the shortest-path metric induced by this tree on the set of its leaves.

- d_T dominates d
- upper bound on d_T: Let u and v be leaves and w be their LCA. Let l_w be the length of the edges from w to its children. Then, $d_T(u, v) \leq 4l_w$.
- Steiner points don't (really) help. (only introducing 4-distortion.) [Gupta, 01; Konjevod *et al.*, 01]

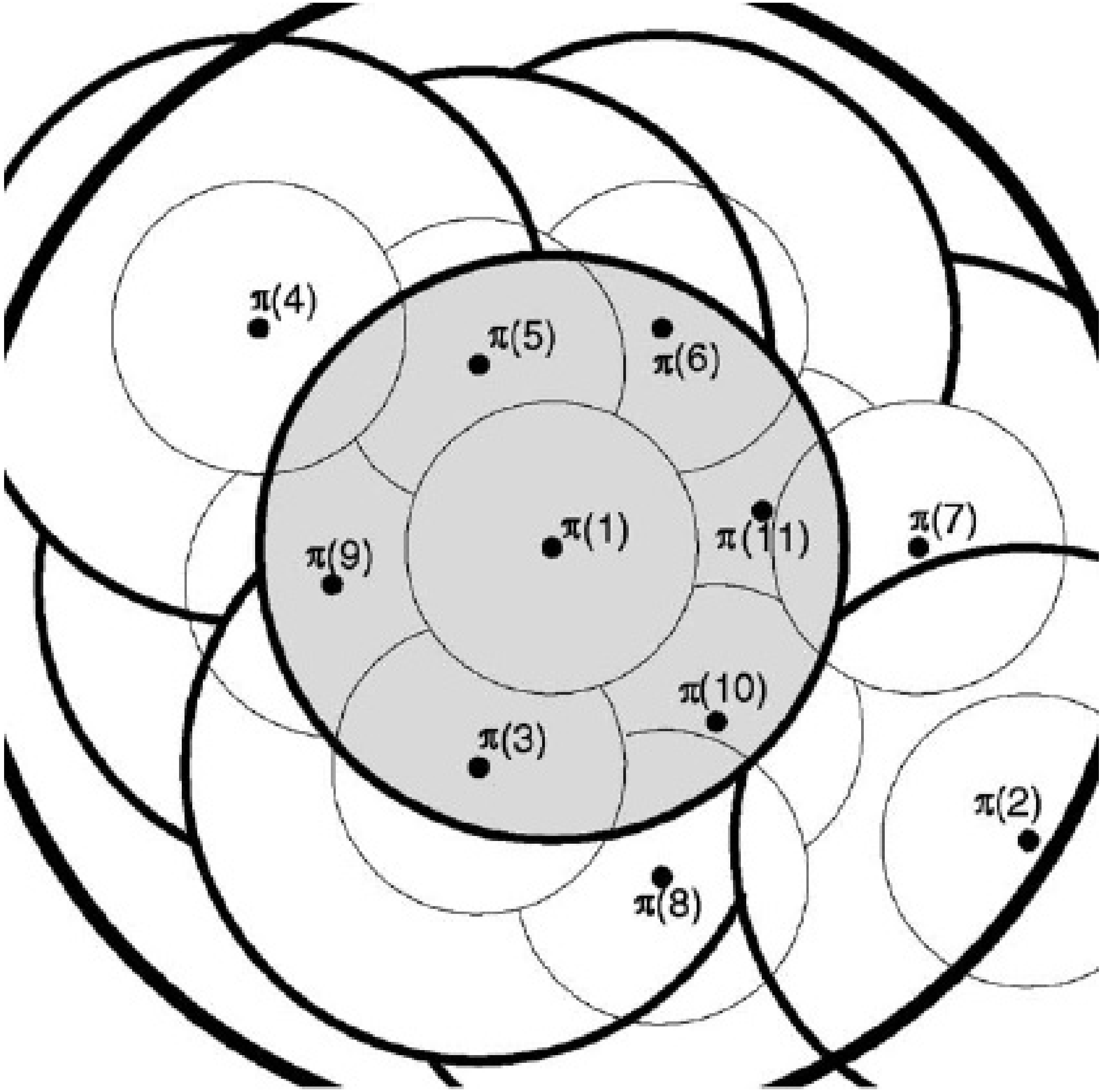
High-Level Plan

- Construct a random hierarchical cut decomposition, and let T be the associated tree
- An edge (u, v) is at level i if u and v are first separated in the decomposition D_i
  - Thus $d_T(u, v) \leq 4 \cdot 2^{i+1} = O(2^i)$
  - Since $d_T(u, v) \geq d(u, v)$, (u, v) cannot be at a level i less than roughly $\log d(u, v)$
  - For i above, we'll show that the probability (u, v) is at level i decreases geometrically with i.
  - $\mathbf{E}[d_T(u, v)] = \sum_i \Pr[(u, v) \text{ is at level } i] \cdot O(2^i)$

Decomposition Algorithm

```
**Algorithm** *Partition* $(V, d)$
Choose a random permutation $\pi$ on $V$.
Choose $R$ uniformly at random from $[\frac{1}{2}, 1]$.
Let $D_\delta = \{V\}$.
for $i = \delta - 1$ downto 0
        Let $R_i = 2^i R$.
        for $l = 1, 2, \ldots, n$
                for every cluster $S \in D_{i+1}$
                        Create a new cluster consisting of all
                        unassigned vertices $v$ in $S$ satisfying
                        $d(\pi(l), v) \leq R_i$
```

Illustration



Analysis

- We get a hierarchical cut decomposition
- Now we only need to prove that given an arbitrary edge (u, v), the expected value of $d_T(u, v)$ is bounded by $O(\log n) \cdot d(u, v)$

- w *settles* the edge (u, v) at level i if w is the first center to which at least one of u and v get assigned at level i.
- Note: exactly one center settles any edge (u, v) at any particular level
- w *cuts* the edge $e = (u, v)$ at level i if it settles e at this level, and exactly one of u and v is assigned to w at level i.

- Define $\mathbf{E}[d_T^w(u, v)] = \sum_i \mathbf{1}(w \text{ cuts } (u, v) \text{ at level } i) \cdot O(2^i)$
- Note:

$$\mathbf{E}[d_T(u, v)] \le \sum_i \Pr[(u, v) \text{ is at level } i] \cdot O(2^i) \le \sum_w \mathbf{E}[d_T^w(u, v)].$$

Analysis cont.

- arrange the points $w_1, w_2, \ldots, w_k, \ldots$ in V in increasing oder of $\min\{d(u, w_k), d(v, w_k)\}$.
- For w_k to cut (u, v),
  - condition A: R_i must fall in $[d(u, w_k), d(v, w_k)]$ for some i. (assume $d(u, w_k) \leq d(v, w_k)$)
  - condition B: w_k settles (u, v) at level i.

- Consider an $x \in [d(u, w_k), d(v, w_k)]$, $\Pr[R_i \text{ falls in } [x, x + dx]] \leq \frac{dx}{2^{i-1}} \leq \frac{2}{x} \cdot dx$
- When A is satisfied, any of $w_1, w_2, \ldots, w_k$ can settle (u, v) at level i. Therefore, $\Pr[B|A] \leq 1/k$
- $\mathbf{E}[d_T^{w_k}(u, v)] \leq \int_{d(u,w_k)}^{d(v,w_k)} \frac{2}{x} \cdot O(x) \cdot \frac{1}{k} \cdot dx = O(\frac{d(v,w_k)-d(u,w_k)}{k}) \leq O(d(u, v)/k)$
- Using linearity of expectation, we have

$$\mathbf{E}[d_T(u, v)] \leq \sum_w \mathbf{E}[d_T^w(u, v)] = \sum_k O(d(u, v)/k) = O(\log n) \cdot d(u, v)$$

Second Analysis

Lemma

Given a vertex u and a radius ρ, the probability that the ball $B(u, \rho)$ is cut at level i is at most $(\rho/2^{i-2}) \cdot \log n$.

- A set S is cut if there are two clusters in the partition such that vertices from S lie in both these components.
- Given an edge $e = (u, v)$, consider the ball of radius $d(e)$ around u. Any partition that cuts the edge e also cuts the ball $B(u, d(e))$.

Proof of Lemma

Proof:

- arrange the points $v_1, v_2, \ldots$ in V in oder of increasing distance from u.
- v_k *intersects* the ball $B(u, \rho)$ if $R_i \in [d(u, v_k) - \rho, d(u, v_k) + \rho]$
- v_k *protects* the ball if $R_i > d(u, v_k) + \rho$
- v_k *cuts the ball first* at level i if,
  - condition A: v_k intersects the ball — $\Pr[A] \leq 2\rho/2^{i-1}$
  - condition B: no node prior to v_k in the permutation π intersects or protects the ball — $\Pr[B|A] \leq 1/k$

$$\begin{aligned}
\Pr[B(u,\rho) \text{ is cut at level } i] &\leq \sum_k \Pr[v_k \text{ cuts } B(u,\rho) \text{ first at level } i] \\
&\leq \sum_k \frac{2\rho}{2^{i-1}} \cdot \frac{1}{k} \\
&\leq (\rho/2^{i-2}) \cdot \log n
\end{aligned}$$

Improvement

Observation

- Since $R_i \in [2^{i-1}, 2^i]$, a node that is closer to u than $2^{i-1} - \rho$ or farther than $2^i + \rho$ cannot cut the ball $B(u, \rho)$ at all.
- we can assume $\rho \leq 2^{i-2}$

$$
\begin{aligned}
\Pr[B(u,\rho) \text{ is cut at level } i] &\leq \sum_{k=|B(u,2^{i-1}-2^{i-2})|}^{|B(u,2^i+r^{i-2})|} \Pr[v_k \text{ cuts } B(u,\rho) \text{ first...}] \\
&\leq \sum_{k=|B(u,2^{i-2})|}^{|B(u,2^{i+1})|} \Pr[v_k \text{ cuts } B(u,\rho) \text{ first at level } i] \\
&\leq (\rho/2^{i-2}) \cdot O\left(\log\left(\frac{|B(u,2^{i+1})|}{|B(u,2^{i-2})|}\right)\right)
\end{aligned}
$$

Final

$$
\begin{aligned}
\mathbf{E}[d_T(u,v)] &\le \sum_i \Pr[(u,v) \text{ is at level } i] \cdot O(2^i) \\
&\le \sum_{i=0}^{\delta-1} O(2^i) \cdot \Pr[(u,v) \text{ is cut at level } i] \\
&\le \sum_{i=0}^{\delta-1} O(2^i) \cdot \Pr[B(u, d(u,v)) \text{ is cut at level } i] \\
&\le \sum_{i=0}^{\delta-1} O(2^i) \cdot \frac{d(u,v)}{2^{i-2}} \cdot O\left(\log\left(\frac{|B(u,2^{i+1})|}{|B(u,2^{i-2})|}\right)\right) \\
&= O(\log n) \cdot d(u,v)
\end{aligned}
$$