The Price of Routing Unsplittable Flow

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Problem Formulation

- Graph $G = (V, E)$ and k source-destination pairs $\{s_i, t_i\}$
- $\mathcal{Q}_i$ denotes the set of (simple) $s_i - t_i$ paths, and
- Latency function $f_e : \mathcal{R}^+ \rightarrow \mathcal{R}^+$
- Bandwidth request (s_j, t_j, w_j) $w_j \in \mathcal{R}^+$
- A flow is a function vector (l_j).

$$l_j : \mathcal{Q}_j \rightarrow \mathcal{R}^+$$

- A flow is feasible if :

$$\sum_{Q \in \mathcal{Q}_j} l_j(Q) = w_j$$

Flow and Strategy

- Splittable Flow

$$l_j(Q) \in [0, w_j]$$

- Unsplittable Flow

$$l_j(Q) \in \{0, w_j\}$$

Pure Strategies:

User j selectes a single path $Q \in \mathcal{Q}_j$.

Mixed Strategies:

User j selectes a probability distribution $\{p_{Q,j}\}$ over $\mathcal{Q}_j$.

Latency for Users

- Pure Strategies:

  Let $\mathcal{S}$ be the system of strategies.

  Let Q_j be the choice of user j, and $\mathcal{Q} = \cup_j Q_j$.

  Define $J(e) = \{j \,|\, e \in \mathcal{Q}\}$ and $l_e = \sum_{j \in J(e)} w_j$.

Latency (per unit) of user j for select path Q (instead of Q_j):

$$c_{Q,j} = \sum_{(e \in Q) \wedge (e \in Q_j)} f_e(l_e) + \sum_{(e \in Q) \wedge (e \notin Q_j)} f_e(l_e + w_j)$$

Latency for Users

- Mixed Strategies:

  Let $\mathcal{S}$ be the system of strategies with $\{p_j\}$

  Let $\{X_{Q,j}\}$ be the set of indicator random variables: whether request j is assigned to Q.

  $X_{e,j} = \sum_{Q|e\in Q} X_{Q,j} \qquad l_e = \sum_{j=1}^{n} X_{e,j} w_j$

Expected latency (per unit) of user j for select path Q in $\mathcal{S}$

$$\begin{aligned} c_{Q,j} &= E[\sum_{e\in Q} f_e(l_e) \,|\, X_{Q,j} = 1] \\ &= E[\sum_{e\in Q} f_e(\sum_{i=1, i\neq j}^{n} X_{e,i} w_i + w_j)] \\ &= \sum_{e\in Q} E[f_e(l_e + (1 - X_{e,j}) w_j)] \end{aligned}$$

Nash Equilibrium

A system $\mathcal{S}$ is at Nash equilibrium if and only if for every $j \in \{1, 2, \ldots, n\}$ and $Q, Q' \in \mathcal{Q}_j$, with $p_{Q,j} > 0 (Q = Q_j)$

$$c_{Q,j} \leq c_{Q',j}$$

Social cost (expected) for system $\mathcal{S}$ is:

$$C(\mathcal{S}) = E[\textstyle\sum_{e \in E} f_e(l_e) l_e]$$

Coordination Ration (Price of Anarchy) is:

$$R = \max_{\mathcal{S}} \frac{C(\mathcal{S})}{C(\mathcal{S}^*)}$$

$\mathcal{S}$ takes over all Nash equilibrium(N.E), and $\mathcal{S}^*$ is the Social Optimal(S.O) solution.

Nash Equilibrium for Linear Latency Functions

Theorem For linear latency functions and pure strategies, the worse-case coordination ratio R is at most $\frac{3+\sqrt{5}}{2} \approx 2.618$

Proof: Let Q_j be the path assigned for request j in N.E. Let Q_j^* be the path assigned for request j in S.O.

$$\begin{aligned}\sum_{e\in Q_j} a_e l_e + b_e &\le \sum_{(e\in Q_j^*)\wedge(e\in Q_j)} a_e l_e + b_e + \sum_{(e\in Q_j^*)\wedge(e\notin Q_j)} a_e(l_e + w_j) + b_e \\ &\le \sum_{e\in Q_j^*} a_e(l_e + w_j) + b_e\end{aligned}$$

$$\sum_{j} \sum_{e\in Q_j} (a_e l_e + b_e) w_j \le \sum_{j} \sum_{e\in Q_j^*} (a_e l_e + b_e) w_j + a_e w_j^2$$

$$\sum_{e\in E} \sum_{j\in J(e)} (a_e l_e + b_e) w_j \le \sum_{e\in E} \sum_{j\in J*(e)} (a_e l_e + b_e) w_j + a_e w_j^2$$

Nash Equilibrium for Linear Latency Functions

Proof (cont'):

$$\sum_{e \in E} \sum_{j \in J(e)} (a_e l_e + b_e) w_j \leq \sum_{e \in E} \sum_{j \in J*(e)} (a_e l_e + b_e) w_j + a_e w_j^2$$

$$\sum_{j \in J(e)} w_j = l_e, \quad \sum_{j \in J*(e)} w_j = l_e^*, \quad \sum_{j \in J*(e)} w_j^d \leq (l_e^*)^d$$

$$\begin{aligned} \sum_{e \in E} (a_e l_e + b_e) l_e &\leq \sum_{e \in E} (a_e l_e + b_e) l_e^* + a_e {l_e^*}^2 \\ &= \sum_{e \in E} a_e l_e l_e^* + \sum_{e \in E} (a_e l_e^* + b_e) l_e^* \end{aligned}$$

Nash Equilibrium for Linear Latency Functions

Proof (cont'):

$$\sum_{e\in E}(a_e l_e + b_e)l_e \leq \sum_{e\in E} a_e l_e l_e^* + \sum_{e\in E}(a_e l_e^* + b_e)l_e^*$$

$$\begin{aligned} \sum_{e\in E} a_e l_e l_e^* &\leq \sqrt{\sum_{e\in E} a_e l_e^2 \sum_{e\in E} a_e {l_e^*}^2} \qquad \text{Cauchy-Schwartz Inequality} \\ &\leq \sqrt{\sum_{e\in E}(a_e l_e + b_e)l_e \sum_{e\in E}(a_e l_e^* + b_e)l_e^*} \end{aligned}$$

$$x = \sqrt{\frac{C(\mathcal{S})}{C(\mathcal{S}^*)}} \qquad\qquad x^2 \leq x + 1,\ \ x^2 \leq \frac{3+\sqrt{5}}{2}$$

Nash Equilibrium for Linear Latency Functions

Unweighted Demand: $w_j = 1$

Theorem For linear latency functions, unweighted demand and pure strategies, the worse-case coordination ratio R is at most 2.5

Proof:

$$\sum_{e \in E} \sum_{j \in J(e)} (a_e l_e + b_e) w_j \leq \sum_{e \in E} \sum_{j \in J^*(e)} (a_e l_e + b_e) w_j + a_e w_j^2$$

$$\sum_{e \in E} (a_e l_e + b_e) l_e \leq \sum_{e \in E} a_e l_e l_e^* + a_e l_e^* + b_e l_e^*$$

Nash Equilibrium for Linear Latency Functions

Proof:

$$\sum_{e\in E}(a_el_e+b_e)l_e \leq \sum_{e\in E} a_el_el_e^* + a_el_e^* + b_el_e^*$$

$$\begin{aligned}
(a_el_e+b_e)l_e &\leq a_el_e^2 + \frac{3}{2}b_el_e = \frac{3}{2}(a_el_e^2+b_el_e) - \frac{1}{2}a_el_e^2 \\
&\leq \frac{3}{2}(al_el_e^* + al_e^* + bl_e^*) - \frac{1}{2}al_e^2 \\
&= \frac{1}{2}a(3l_el_e^* + 3l_e^* - l_e^2) + \frac{3}{2}b_el_e^* \\
&\leq \frac{5}{2}a_el_e^{*2} + \frac{3}{2}b_el_e^* \qquad 3ij + 3j - i^2 \leq 5j^2 \\
&\leq \frac{5}{2}(a_el_e^* + b_e)l_e^*
\end{aligned}$$

Nash Equilibrium for Linear Latency Functions

Theorem For linear latency functions and mixed strategies, the worse-case coordination ratio R is at most $\frac{3+\sqrt{5}}{2} \approx 2.618$

Proof:

$$\begin{aligned} c_{Q,j} &= E[\sum_{e\in Q} f_e(l_e) \,|\, X_{Q,j} = 1] \\ &= E[\sum_{e\in Q} f_e(\sum_{i=1,i\neq j}^{n} X_{e,i} w_i + w_j)] \\ &= \sum_{e\in Q} E[f_e(l_e + (1 - X_{e,j})w_j)] \end{aligned}$$

The change from $X_{Q,j}$ to $X_{e,j}$ does not affect the proofs. In particular, the proof of **Lemma 3.4** is still correct, if we replace $p_{Q,j} - p_{Q,j}^2$ by $(1 - p_{e,j})p_{Q,j}$.

Nash Equilibrium for Linear Latency Functions

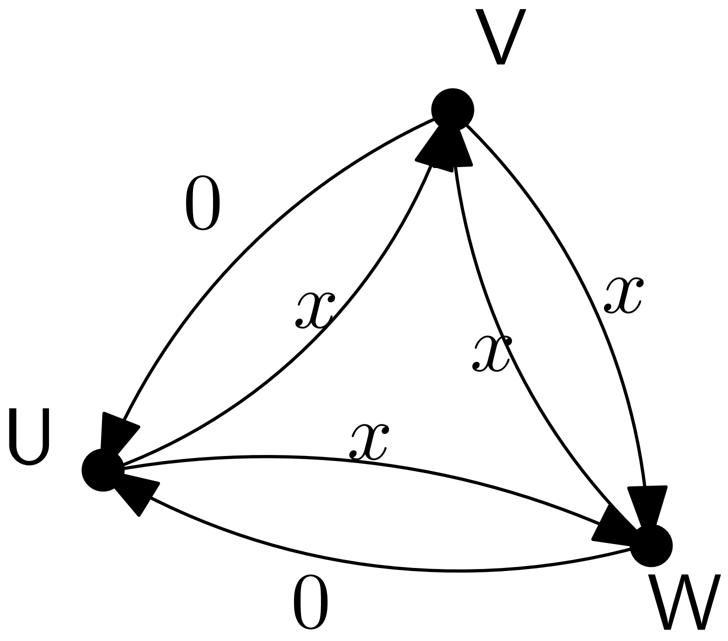
Remarks:

If we allow splittable flows, the price of anarchy is bounded by $\frac{4}{3}$ [Roughgarden, SODA 05]

> Though I am doubt on this result, as the Proposition 1 there is counter intuitive to me.

Unweighted demand will not achieve better ratio in mixed strategies. Because we lose the properties for integers.

Lower Bounds for Linear Latency Functions



Demands: $\phi = \frac{1+\sqrt{5}}{2}, 1$

- User 1: (U, V, ϕ)
- User 2: (U, W, ϕ)
- User 3: $(V, W, 1)$
- User 4: $(W, V, 1)$

Optimal: $2\phi^2 + 2$

- User 1: UV
- User 2: UW
- User 3: VW
- User 4: WV

N.E $2\phi^2 + 2(\phi + 1)^2$

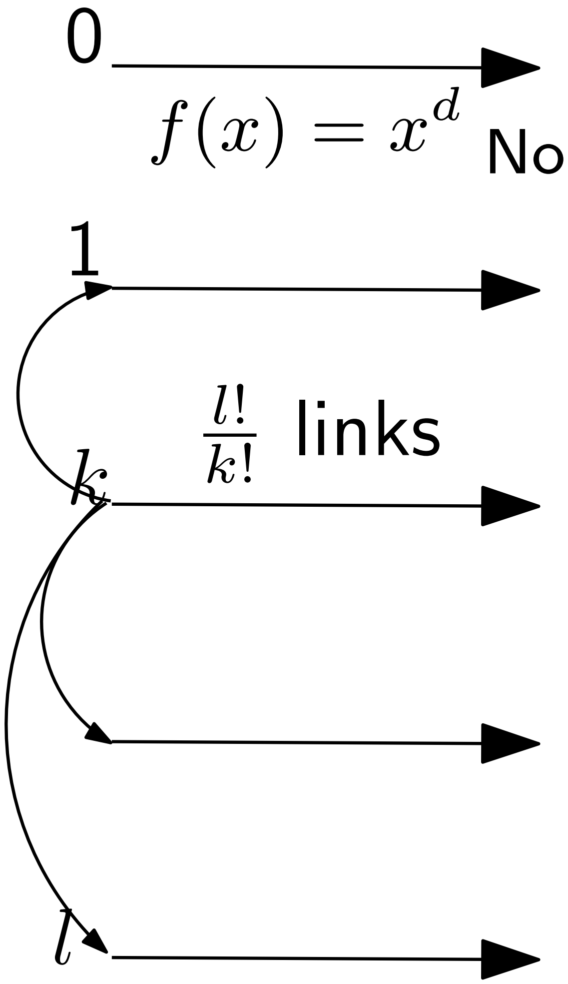
- User 1: UWV
- User 2: UVW
- User 3: VUW
- User 4: WUV

Nash Equilibrium for Polynomial Latency Functions

Theorem For polynomial latency functions of degree d and pure and mixed strategies, the worse-case coordination ratio R is $O(2^d d^{d+1})$

Theorem For polynomial latency functions of degree d and pure strategies, the worse-case coordination ratio R is $\Omega(d^{d/2})$

Lower Bounds for Polynomial Latency Functions



No job for grup 0

$\frac{l!}{k-1!}$ jobs

Optimal:

Group k assigns jobs to links of group $k-1$.

Nash Equilibrium:

Group k assigns jobs to links of group k.

$$OPT = \sum_{k=0}^{l-1} \frac{l!}{k!} 1^d = l! \sum_{k=0}^{l-1} \frac{1}{k!} \approx l! \cdot e$$

$$NE = \sum_{k=1}^{l} \frac{l!}{k!} k^d \geq \frac{l!}{(d/2)^d} \cdot (d/2)^d = l! \cdot \Omega(d^{d/2})$$

Remaining:

Lower bounds for mixed strategies.

Gap in the bounds of polynomial latency functions: $O(2^d d^{d+1})$ and $\Omega(d^{d/2})$.