How Bad is Selfish Routing

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Problem Formulation: Traffic Model

- Given the rate of traffic between each pair of nodes in a network, find an assignment of traffic to minimize the total latency.

- On each edge, the latency is load dependent.

- Each player controls a negligible fraction of the overall traffic.

\[ l(x) = x \]
\[ l(x) = 1 \]
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Braess’s Paradox
Formal Model

- Graph $G = (V, E)$ and $k$ source-destination pairs $\{s_i, t_i\}$
- $P_i$ denotes the set of (simple) $s_i - t_i$ paths, and
- $P = \bigcup_i P_i$
- A flow is a function:
  $$f : P \rightarrow \mathcal{R}^+$$
- A flow is feasible if:
  $$\sum_{P \in P_i} f_P = r_i$$
- Each edge has a nonnegative, differentiable, nondecreasing latency function $l_e(\cdot)$
Cost for Flows

- Let \((G, r, l)\) be an instance, and \(f\) is a flow.

  \[ f_e = \sum_{P: e \in P} f_P \]

- Latency of a path \(P\)

  \[ l_P(f) = \sum_{e \in P} l_e(f_e) \]

- Cost of a flow \(f\):

  \[ C(f) = \sum_{P \in P} l_P(f) f_P = \sum_{e \in E} l_e(f_e) f_e \]

- Players are small flows behave "greedily" and "selfishly"

  There are infinite number of players, each carry a negligible amount of flow.
Flows at Nash Equilibrium

• **Definition (Nash Equilibrium):**

A flow $f$ is feasible for instance $(G, r, l)$ is at Nash Equilibrium if for all $i \in \{1, \ldots, k\}$, $P_1, P_2 \in P_i$, and $\delta \in [0, f_{P_1}]$, we have $l_{P_1}(f) \leq l_{P_2}(\tilde{f})$, where

$$\tilde{f}_P = \begin{cases} 
  f_P - \delta & \text{if } P = P_1 \\
  f_P + \delta & \text{if } P = P_2 \\
  f_P & \text{if } P \notin \{P_1, P_2\}
\end{cases}$$

• **Lemma:** A flow $f$ feasible for instance $(G, r, l)$ is at Nash Equilibrium if and only if for all $i \in \{1, \ldots, k\}$, $P_1, P_2 \in P_i$ with $f_{P_1} > 0$, $l_{P_1}(f) \leq l_{P_2}(f)$. 
Optimal Flows via Convex Programming

- NonLinear Programming Formulation

Min $\sum_{e \in E} c_e(f_e)$

subject to:

$\sum_{P \in \mathcal{P}_i} f_P = r_i \quad \forall i \in \{1, \ldots, k\}$

$f_e = \sum_{P \in \mathcal{P} : e \in P} f_P \quad \forall e \in E$

$f_P \geq 0 \quad \forall P \in \mathcal{P}$
Characteristic of Optimal Flows

Let $c'_e$ be the derivative $\frac{d}{dx} c_e(x)$

$$c'_P(f) = \sum_{e \in P} c'_e(f_e)$$

- **Lemma**: A flow $f$ is optimal for a convex program of the previous form if and only if for every $i \in \{1, \ldots, k\}$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $c'_{P_1}(f) \leq c'_{P_2}(f)$. 


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• **Lemma:** A flow \( f \) feasible for instance \((G, r, l)\) is at Nash Equilibrium if and only if for all \( i \in \{1, \ldots, k\} \), \( P_1, P_2 \in \mathcal{P}_i \) with \( f_{P_1} > 0 \), \( l_{P_1}(f) \leq l_{P_2}(f) \).

\[
C(f) = \sum_{i=1}^{k} L_i(f)r_i
\]
Nash Equilibrium and Optimal Flow

Marginal cost function:

\[ l^*_e(f_e) = (l_e(f_e)f_e)' = l_e(f_e) + l'_e(f_e)f_e \]

- **Corollary:** Let \((G, r, l)\) be an instance in which \(x \cdot l_e(x)\) is a convex function for each edge \(e\), with marginal cost functions \(l^*_e\). Then a flow \(f\) feasible for \((G, r, l)\) is optimal if and only if it is at Nash equilibrium for the instance \((G, r, l^*)\)
Nash Equilibrium and Optimal Flow (cont’)

• **Lemma:** An instance \((G, r, l)\) with continuous, nondecreasing latency functions admits a feasible flow at Nash equilibrium. Moreover, if \(f, \tilde{f}\) are flows at Nash equilibrium, then \(C(f) = C(\tilde{f})\).

**Proof:** Set \(h_e(x) = \int_0^x l_e(t)dt\)

\[
\begin{align*}
\text{Min} & \quad \sum_{e \in E} h_e(f_e) \\
\sum_{P \in \mathcal{P}_i} f_P &= r_i \quad \forall i \in \{1, \ldots, k\} \\
f_e &= \sum_{P \in \mathcal{P}: e \in P} f_P \quad \forall e \in E \\
\text{Note,} & \quad h_e'(x) = l_e(x) \quad f_P \geq 0 \quad \forall P \in \mathcal{P}
\end{align*}
\]
"Unique" Nash Equilibrium

- **Lemma:** An instance \((G, r, l)\) with continuous, nondecreasing latency functions admits a feasible flow at Nash equilibrium. Moreover, if \(f, \tilde{f}\) are flows at Nash equilibrium, then \(C(f) = C(\tilde{f})\).

**Proof (cont’):**

Set \(h_e(x) = \int_0^x l_e(t)dt\)

Min \(\sum_{e \in E} h_e(f_e)\)

If \(f_e \neq \tilde{f}_e\), the function \(h_e(x)\) must be linear and \(l_e\) is a constant function

This implies \(l_e(f_e) = l_e(\tilde{f}_e)\).

\[C(f) = \sum_{i=1}^{k} L_i(f) r_i = C(\tilde{f}).\]
Nontrivial Upper Bound for Price of Anarchy

For instance \((G, r, l)\), let \(f^*\) be an optimal flow and \(f\) be a flow at Nash equilibrium.

\[
\rho = \rho(G, r, l) = \frac{C(f)}{C(f^*)}
\]

Corollary: Suppose the instance \((G, r, l)\) and the constant \(\alpha \geq 1\) satisfy:

\[
x \cdot l_e(x) \leq \alpha \cdot \int_0^x l_e(t) dt
\]

\[
\rho(G, r, l) \leq \alpha
\]
Nontrivial Upper Bound for Price of Anarchy (cont’)

**Corollary:** Suppose the instance \((G, r, l)\) and the constant \(\alpha \geq 1\) satisfy:

\[
x \cdot l_e(x) \leq \alpha \cdot \int_0^x l_e(t)dt
\]

\[
\rho(G, r, l) \leq \alpha
\]

**Proof:**

\[
C(f) = \sum_{e \in E} l_e(f_e)f_e
\]

\[
\leq \alpha \sum_{e \in E} \int_0^{f_e} l_e(t)dt
\]

\[
\leq \alpha \sum_{e \in E} \int_0^{f_e^*} l_e(t)dt
\]

\[
\leq \alpha \sum_{e \in E} l_e(f_e^*)f_e^*
\]

\[
= \alpha \cdot C(f^*)
\]
Upper Bound for Polynomial Latency Function

**Corollary:** Suppose the instance \((G, r, l)\) has the latency functions:

\[
l_e(x) = \sum_{i=0}^{p} a_{e,i}x^i \quad a_{e,i} \geq 0
\]

\[
\rho(G, r, l) \leq p + 1
\]

**Remarks:** It is not tight.

\[
l_e(x) = a_e x + b_e \quad \text{for } a_e, b_e \geq 0 \quad \rho \leq 2
\]

Tight Bound: \(\rho \leq 4/3\)

For higher degree polynomial latency functions:

\[
\rho = O\left(\frac{p}{\ln p}\right)
\]
A Bicriteria Result for General Latency Functions

Negative Result: 

If \( l(x) = x^p \): **Optimal** flows assigns \((p + 1)^{-1/p}\) on the lower link, which has a total latency:

\[
1 - p(p + 1)^{-(p+1)/p} \to 0
\]

\( \rho \to \infty \)
Augment Analysis for General Latency Function

- **Theorem:** If \( f \) is a flow at Nash equilibrium for \((G, r, l)\) and \( f^* \) is feasible for \((G, 2r, l)\), then \( C(f) \leq C(f^*) \)

Let

\[
\bar{l}_e(x) = \begin{cases} 
  l_e(f_e) & \text{if } x \leq f_e \\
  l_e(x) & \text{if } x \geq f_e 
\end{cases}
\]

\[
\sum_e \bar{l}_e(f^*_e)f^*_e - C(f^*) = \sum_{e \in E} f^*_e(\bar{l}_e(f^*_e) - l_e(f^*_e)) \leq \sum_{e \in E} l_e(f_e)f_e = C(f)
\]

\[
\bar{l}_P(f^*) \geq \bar{l}_P(f_0) \geq L_i(f) \geq L_i(f) \geq L_i(f)
\]

\[
\sum_e \bar{l}_P(f^*_e)f^*_P \geq \sum_i \sum_{P \in P_i} L_i(f)f^*_P = \sum_i 2L_i(f)r_i = 2C(f)
\]
Worst-Case Ratio with Linear Latency Functions

\[ l_e = a_e x + b_e \text{ with } a_e, b_e \geq 0 \]

\[ l_e^* = 2a_e x + b_e \]

- **Lemma:** If \((G, r, l)\) be an instance with edge latency functions \(l_e(x) = a_e x + b_e\) for each edge \(e \in E\). Then

  (a) a flow \(f\) is at **Nash equilibrium** in \(G\) if and only if for \(P, P' \in \mathcal{P}_i\) with \(f_P > 0\),

  \[ \sum_{e \in P} a_e f_e + b_e \leq \sum_{e \in P'} a_e f_e + b_e \]

  (b) a flow \(f^*\) is (globally) **Optimal** in \(G\) if and only if for \(P, P' \in \mathcal{P}_i\) with \(f^*_P > 0\),

  \[ \sum_{e \in P} 2a_e f^*_e + b_e \leq \sum_{e \in P'} 2a_e f^*_e + b_e \]
Worst-Case Ratio with Linear Latency Functions (cont’)

- **Lemma:** Suppose \((G, r, l)\) has linear latency functions and \(f\) is a flow at Nash equilibrium. Then

  (a) The flow \(f/2\) is optimal for \((G, r/2, l)\)

  (b) the marginal cost of increasing the flow on a path \(P\) for \(f/2\) equals the latency of \(P\) for \(f\)

\[
l_P^*(f/2) = l_P(f)
\]

Creating optimal flow in two steps: \((f\) is at Nash equilibrium\)

1. Send a flow optimal for instance \((G, r/2, l)\). \(C(f)/4\)
2. Augment to one optimal for instance \((G, r, l)\). \(C(f)/2\)
Augment Cost for Linear Latency Functions

- **Lemma**: \((G, r, l)\) has linear latency functions and \(f^*\) is an optimal flow. Let \(L_i^*(f^*)\) be the minimum marginal cost for \(s_i - t_i\) paths. For any \(\delta > 0\), a feasible flow \(f\) for 
\((G, (1 + \delta)r, l)\):

\[
C(f) \geq C(f^*) + \delta \sum_{i=1}^{k} L_i^*(f^*) r_i
\]

\[x \cdot l_e(x) = a_e x^2 + b_e\] is convex.

\[
l_e(f_e)f_e \geq l_e(f_e^*)f^* + (f_e - f^*)l_e(f_e^*)
\]
Augment Cost for Linear Latency Functions

Proof:

\[
C(f) = \sum_{e \in E} l_e(f_e)f_e \\
\geq \sum_{e \in E} l_e(f^*_e)f^*_e + \sum_{e \in E} (f_e - f^*_e)l^*_e(f^*_e) \\
= C(f^*) + \sum_{i=1}^{k} \sum_{P \in P_i} l^*_P(f^*)(f_P - f^*_P) \\
\geq C(f^*) + \sum_{i=1}^{k} L^*_i(f^*) \sum_{P \in P_i} (f_P - f^*_P) \\
= C(f^*) + \delta \sum_{i=1}^{k} L^*_i(f^*)r_i
\]
Worst-Case Ratio with Linear Latency Functions (cont’)

- **Lemma:** If \((G, r, l)\) has linear latency functions, then 
  \[ \rho(G, r, l) \leq \frac{4}{3} \]

**Proof:** Let \(f\) be a flow at N.E. \(f/2\) is optimal for \((G, r/2, l)\). Moreover, \(L^*_i(f/2) = L_i(f)\).

\[
C(f^*) \geq C(f/2) + \sum_{i=1}^{k} L^*_i(f/2) \frac{r_i}{2} \\
= C(f/2) + \frac{1}{2} \sum_{i=1}^{k} L_i(f) r_i \\
= C(f/2) + \frac{1}{2} C(f) \\
\geq \frac{3}{4} C(f)
\]

\[
C(f/2) = \frac{1}{4} a_educ^2 + \frac{1}{2} b_epe^e \\
\geq \frac{1}{4} \sum_e (a_educ^2 + b_epe^e) \\
= \frac{1}{4} C(f)
\]
Extensions:

- **Approximate Nash Equilibrium:**
  
  If \( f \) is at \( \epsilon \) N.E, and \( f^* \) is feasible for \((G, 2r, l)\), then 
  \[
  C(f) \leq \frac{1 + \epsilon}{1 - \epsilon} C(f^*).
  \]

- **Finite Agents: Splittable Flow**
  
  \[
  C(f) \leq C(f^*).
  \]

- **Finite Agents: Unsplittable Flow**
  
  If for some \( \alpha < 2 \), 
  \[
  l_e(x + r_i) \leq \alpha \cdot l_e(x), \quad x \in [0, \sum_{j \neq i} r_j]
  \]
  \[
  C(f) \leq \frac{\alpha}{2 - \alpha} C(f^*).
  \]