

How Bad is Selfish Routing

Tim Roughgarden

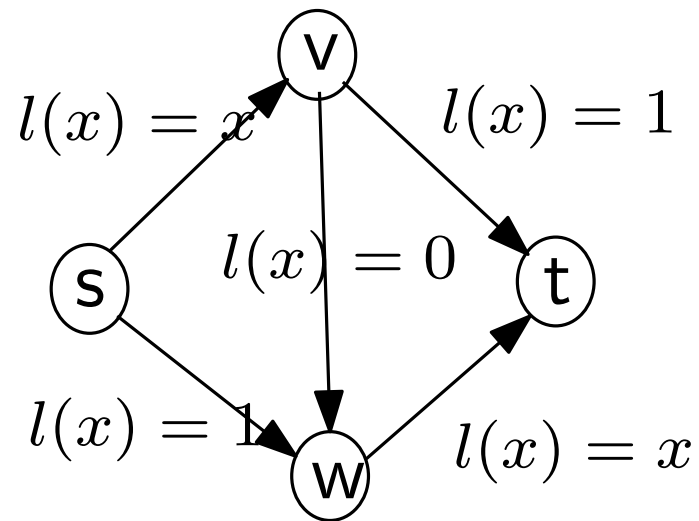
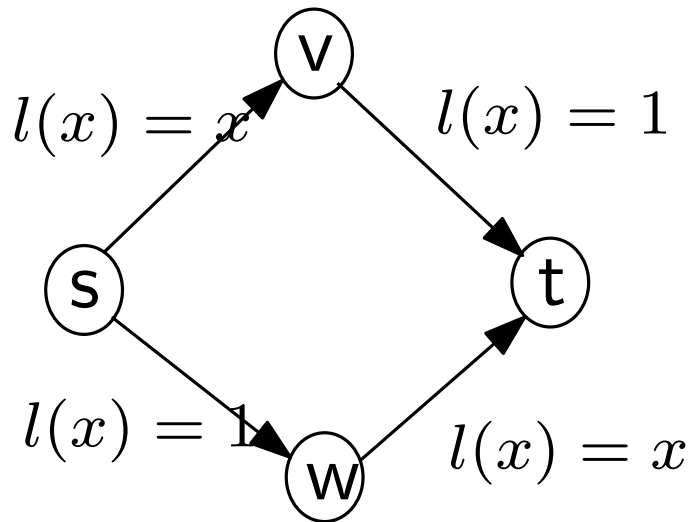
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Problem Formulation: Traffic Model

- Given the rate of traffic between each pair of nodes in a network, find an assignment of traffic to **minimize** the **total latency**.
- On each edge, the **latency** is **load dependent**
- **Each player** controls a **negligible** fraction of the overall traffic.



Braess's Paradox

Formal Model

- Graph $G = (V, E)$ and k source-destination pairs $\{s_i, t_i\}$
- \mathcal{P}_i denotes the set of (simple) $s_i - t_i$ paths, and
- $\mathcal{P} = \cup_i \mathcal{P}_i$

- A flow is a function:

$$f : \mathcal{P} \rightarrow \mathcal{R}^+$$

- A flow is feasible if :

$$\sum_{P \in \mathcal{P}_i} f_P = r_i$$

- Each edge has a nonnegative, differentiable, nondecreasing latency function $l_e(\cdot)$

Cost for Flows

- Let (G, r, l) be an **instance** , and f is a **flow**.

$$f_e = \sum_{P:e \in P} f_P$$

- Latency of a path P

$$l_P(f) = \sum_{e \in P} l_e(f_e)$$

- Cost of a flow f :

$$C(f) = \sum_{P \in \mathcal{P}} l_P(f) f_P = \sum_{e \in E} l_e(f_e) f_e$$

- **Players** are **small flows** behave "greedily" and "selfishly"

There are **infinite** number of players, each carry a **negligible** amount of flow.

Flows at Nash Equilibrium

- **Definition (Nash Equilibrium):**

A flow f is feasible for instance (G, r, l) is at **Nash Equilibrium** if for all $i \in \{1, \dots, k\}$, $P_1, P_2 \in \mathcal{P}_i$, and $\delta \in [0, f_{P_1}]$, we have $l_{P_1}(f) \leq l_{P_2}(\tilde{f})$, where

$$\tilde{f}_P = \begin{cases} f_P - \delta & \text{if } P = P_1 \\ f_P + \delta & \text{if } P = P_2 \\ f_P & \text{if } P \notin \{P_1, P_2\} \end{cases}$$

- **Lemma:** A flow f feasible for instance (G, r, l) is at **Nash Equilibrium** if and only if for all $i \in \{1, \dots, k\}$, $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $l_{P_1}(f) \leq l_{P_2}(f)$.

Optimal Flows via Convex Programming

- NonLinear Programming Formulation

$$\text{Min } \sum_{e \in E} c_e(f_e)$$

subject to:

$$\sum_{P \in \mathcal{P}_i} f_P = r_i \quad \forall i \in \{1, \dots, k\}$$

$$f_e = \sum_{P \in \mathcal{P}: e \in P} f_P \quad \forall e \in E$$

$$f_P \geq 0 \quad \forall P \in \mathcal{P}$$

Characteristic of Optimal Flows

Let c'_e be the derivative $\frac{d}{dx} c_e(x)$

$$c'_P(f) = \sum_{e \in P} c'_e(f_e)$$

- **Lemma:** A **flow** f is **optimal** for a convex program of the previous form if and only if for every $i \in \{1, \dots, k\}$ and $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $c'_{P_1}(f) \leq c'_{P_2}(f)$.

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- **Lemma:** A **flow** f feasible for instance (G, r, l) is at **Nash Equilibrium** if and only if for all $i \in \{1, \dots, k\}$, $P_1, P_2 \in \mathcal{P}_i$ with $f_{P_1} > 0$, $l_{P_1}(f) \leq l_{P_2}(f)$.

$$C(f) = \sum_{i=1}^k L_i(f)r_i$$

Nash Equilibrium and Optimal Flow

Marginal cost function:

$$l_e^*(f_e) = (l_e(f_e)f_e)' = l_e(f_e) + l'_e(f_e)f_e$$

- **Corollary:** Let (G, r, l) be an instance in which $x \cdot l_e(x)$ is a convex function for each edge e , with marginal cost functions l_e^* . Then a flow f **feasible** for (G, r, l) is **optimal** if and only if it is at **Nash equilibrium** for the instance (G, r, l^*)

Nash Equilibrium and Optimal Flow (cont')

- **Lemma:** An instance (G, r, l) with continuous, nondecreasing latency functions admits a feasible flow at **Nash equilibrium**. Moreover, if f, \tilde{f} are flows at **Nash equilibrium**, then $C(f) = C(\tilde{f})$.

Proof: Set $h_e(x) = \int_0^x l_e(t) dt$

$$\begin{array}{ll} \text{Min } \sum_{e \in E} h_e(f_e) & \\ & \sum_{P \in \mathcal{P}_i} f_P = r_i \quad \forall i \in \{1, \dots, k\} \\ & f_e = \sum_{P \in \mathcal{P}: e \in P} f_P \quad \forall e \in E \end{array}$$

$$\text{Note, } h'_e(x) = l_e(x) \quad f_P \geq 0 \quad \forall P \in \mathcal{P}$$

"Unique" Nash Equilibrium

- **Lemma:** An instance (G, r, l) with continuous, nondecreasing latency functions admits a feasible flow at **Nash equilibrium**. Moreover, if f, \tilde{f} are flows at **Nash equilibrium**, then $C(f) = C(\tilde{f})$.

Proof (cont') :

$$\text{Set } h_e(x) = \int_0^x l_e(t) dt$$

If $f_e \neq \tilde{f}_e$, the function $h_e(x)$ must be linear and l_e is a constant function

$$\text{Min } \sum_{e \in E} h_e(f_e)$$

This implies $l_e(f_e) = l_e(\tilde{f}_e)$.

$$C(f) = \sum_{i=1}^k L_i(f) r_i = C(\tilde{f}).$$

Nontrivial Upper Bound for Price of Anarchy

For instance (G, r, l) , let f^* be an **optimal** flow and f be a flow at **Nash equilibrium**.

$$\rho = \rho(G, r, l) = \frac{C(f)}{C(f^*)}$$

Corollary: Suppose the instance (G, r, l) and the constant $\alpha \geq 1$ satisfy:

$$x \cdot l_e(x) \leq \alpha \cdot \int_0^x l_e(t) dt$$

$$\rho(G, r, l) \leq \alpha$$

Nontrivial Upper Bound for Price of Anarchy (cont')

Corollary: Suppose the instance (G, r, l) and the constant $\alpha \geq 1$ satisfy:

$$x \cdot l_e(x) \leq \alpha \cdot \int_0^x l_e(t) dt$$

$$\rho(G, r, l) \leq \alpha$$

Proof:

$$\begin{aligned} C(f) &= \sum_{e \in E} l_e(f_e) f_e \\ &\leq \alpha \sum_{e \in E} \int_0^{f_e} l_e(t) dt \\ &\leq \alpha \sum_{e \in E} \int_0^{f_e^*} l_e(t) dt \\ &\leq \alpha \sum_{e \in E} l_e(f_e^*) f_e^* \\ &= \alpha \cdot C(f^*) \end{aligned}$$

N.E optimizes this objective function.

Upper Bound for Polynomial Latency Function

Corollary: Suppose the instance (G, r, l) has the latency functions:

$$l_e(x) = \sum_{i=0}^p a_{e,i} x^i \quad a_{e,i} \geq 0$$

$$\rho(G, r, l) \leq p + 1$$

Remarks: It is not tight.

$$l_e(x) = a_e x + b_e \text{ for } a_e, b_e \geq 0 \quad \rho \leq 2$$

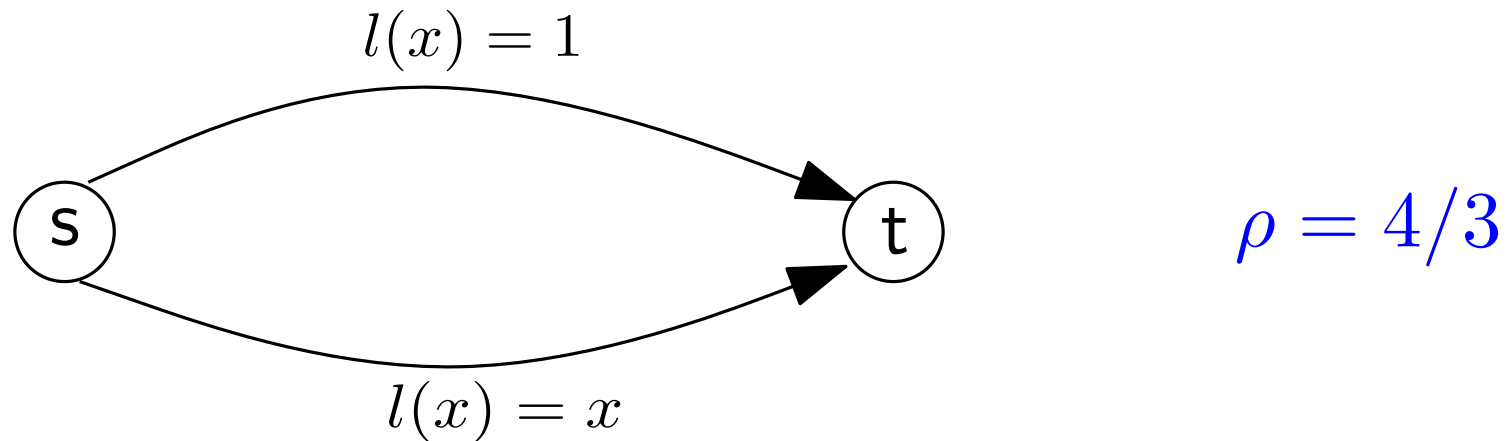
Tight Bound: $\rho \leq 4/3$

For higher degree **polynomial latency functions:**

$$\rho = O\left(\frac{p}{\ln p}\right)$$

A Bicriteria Result for General Latency Functions

Negative Result :



If $l(x) = x^p$: **Optimal** flows assigns $(p+1)^{-1/p}$ on the lower link, which has a total latency:

$$1 - p(p+1)^{-(p+1)/p} \rightarrow 0$$

$$\rho \rightarrow \infty$$

Augment Analysis for General Latency Function

- **Theorem:** If f is a flow at **Nash equilibrium** for (G, r, l) and f^* is feasible for $(G, 2r, l)$, then $C(f) \leq C(f^*)$

Let

$$\bar{l}_e(x) = \begin{cases} l_e(f_e) & \text{if } x \leq f_e \\ l_e(x) & \text{if } x \geq f_e \end{cases}$$

$$\begin{aligned} \sum_e \bar{l}_e(f_e^*) f_e^* - C(f^*) &= \sum_{e \in E} f_e^* (\bar{l}_e(f_e^*) - l_e(f_e^*)) \\ &\leq \sum_{e \in E} l_e(f_e) f_e \\ &= C(f) \end{aligned}$$

$$\begin{aligned} \bar{l}_P(f^*) \geq \bar{l}_P(f_0) \geq L_i(f) \quad \sum_e \bar{l}_P(f^*) f_P^* &\geq \sum_i \sum_{P \in \mathcal{P}_i} L_i(f) f_P^* \\ &= \sum_i 2L_i(f) r_i \\ &= 2C(f) \end{aligned}$$

Worst-Case Ratio with Linear Latency Functions

$$l_e = a_e x + b_e \text{ with } a_e, b_e \geq 0$$

$$l_e^* = 2a_e x + b_e$$

- **Lemma:** If (G, r, l) be an instance with edge latency functions $l_e(x) = a_e x + b_e$ for each edge $e \in E$. Then

(a) a flow f is at **Nash equilibrium** in G if and only if for $P, P' \in \mathcal{P}_i$ with $f_P > 0$,

$$\sum_{e \in P} a_e f_e + b_e \leq \sum_{e \in P'} a_e f_e + b_e$$

(b) a flow f^* is (globally) **Optimal** in G if and only if for $P, P' \in \mathcal{P}_i$ with $f_P^* > 0$,

$$\sum_{e \in P} 2a_e f_e^* + b_e \leq \sum_{e \in P'} 2a_e f_e^* + b_e$$

Worst-Case Ratio with Linear Latency Functions (cont')

- **Lemma:** Suppose (G, r, l) has linear latency functions and f is a flow at **Nash equilibrium**. Then
 - (a) The flow $f/2$ is **optimal** for $(G, r/2, l)$
 - (b) the **marginal cost** of increasing the flow on a path P for $f/2$ equals the **latency** of P for f

$$l_P^*(f/2) = l_P(f)$$

Creating **optimal** flow in **two** steps: (f is at **Nash equilibrium**)

- (1) Send a flow **optimal** for instance $(G, r/2, l)$. $C(f)/4$
- (2) Augment to one **optimal** for instance (G, r, l) . $C(f)/2$

Augment Cost for Linear Latency Functions

- **Lemma:** (G, r, l) has linear latency functions and f^* is an **optimal** flow. Let $L_i^*(f^*)$ be the minimum marginal cost for $s_i - t_i$ paths. For any $\delta > 0$, a feasible flow f for $(G, (1 + \delta)r, l)$:

$$C(f) \geq C(f^*) + \delta \sum_{i=1}^k L_i^*(f^*) r_i$$

$x \cdot l_e(x) = a_e x^2 + b_e$ is convex.

$$l_e(f_e) f_e \geq l_e(f_e^*) f_e^* + (f_e - f_e^*) l_e^*(f_e^*)$$

Augment Cost for Linear Latency Functions

- Proof:

$$\begin{aligned} C(f) &= \sum_{e \in E} l_e(f_e) f_e \\ &\geq \sum_{e \in E} l_e(f_e^*) f_e^* + \sum_{e \in E} (f_e - f_e^*) l_e^*(f_e^*) \\ &= C(f^*) + \sum_{i=1}^k \sum_{P \in \mathcal{P}_i} l_P^*(f^*) (f_P - f_P^*) \\ &\geq C(f^*) + \sum_{i=1}^k L_i^*(f^*) \sum_{P \in \mathcal{P}_i} (f_P - f_P^*) \\ &= C(f^*) + \delta \sum_{i=1}^k L_i^*(f^*) r_i \end{aligned}$$

Worst-Case Ratio with Linear Latency Functions (cont')

- **Lemma:** If (G, r, l) has linear latency functions, then $\rho(G, r, l) \leq 4/3$

Proof: Let f be a flow at N.E. $f/2$ is **optimal** for $(G, r/2, l)$. Moreover, $L_i^*(f/2) = L_i(f)$.

$$\begin{aligned} C(f^*) &\geq C(f/2) + \sum_{i=1}^k L_i^*(f/2) \frac{r_i}{2} & C(f/2) &= \frac{1}{4} a_e f_e^2 + \frac{1}{2} b_e f_e \\ &= C(f/2) + \frac{1}{2} \sum_{i=1}^k L_i(f) r_i & &\geq \frac{1}{4} \sum_e (a_e f_e^2 + b_e f_e) \\ &= C(f/2) + \frac{1}{2} C(f) & &= \frac{1}{4} C(f) \\ &\geq \frac{3}{4} C(f) \end{aligned}$$

Extensions:

- Approximate Nash Equilibrium:

If f is at ϵ **N.E**, and f^* is feasible for $(G, 2r, l)$, then
$$C(f) \leq \frac{1+\epsilon}{1-\epsilon} C(f^*).$$

- Finite Agents: Splittable Flow

$$C(f) \leq C(f^*).$$

- Finite Agents: Unsplittable Flow

If for some $\alpha < 2$, $l_e(x + r_i) \leq \alpha \cdot l_e(x)$, $x \in [0, \sum_{j \neq i} r_j]$

$$C(f) \leq \frac{\alpha}{2-\alpha} C(f^*).$$