Competitive Analysis of Incentive Compatible On-line Auctions

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Outline

- The On-line Auction Model
- Incentive & Supply Curves
- Terminologies
  - Global Supply Curve
  - Revenue & Social Efficiency
  - Off-line Vickrey Auction
  - Competitiveness
- One Divisible Good
- *k* Indivisible Goods
  - A Randomized Auction
  - A Deterministic Auction
  - Revenue Analysis on Uniform Distribution

The Model

- **The goods**

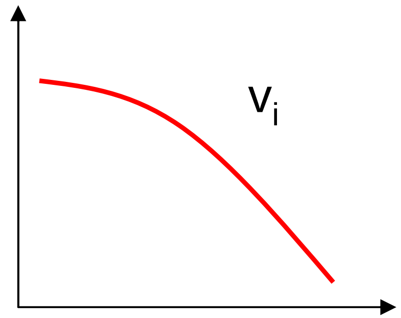
  k identical indivisible goods

  when k is very large $\rightarrow$ one divisible good

- **Players' valuations and utilities**

Player i has marginal valuation $v_i(j)$ for good j, $1 \leq j \leq k$

Assume that $\forall$ i, j: $v_i(j+1) \leq v_i(j)$



When player i receives q goods and pays P_i

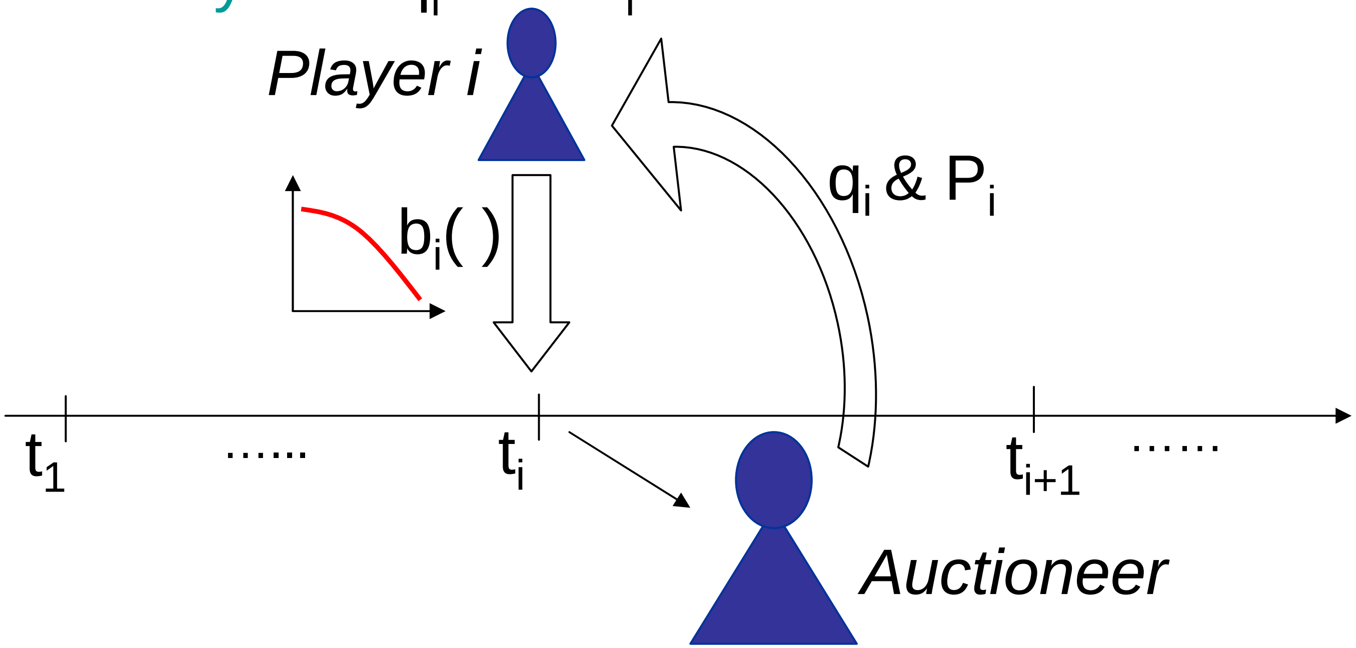
his utility is $U_i(q, P_i) = \sum_{j=1}^{q} v_i(j) - P_i$

Each player aims to maximize his utility

The Model

- **The on-line game and players' strategies**

At time t_i, player i declares function $b_i(\cdot)$ as his marginal function $b_i : [1\ldots k] \to R$, non-increasing (of coz he could lie, i.e. $b_i(\cdot) \neq v_i(\cdot)$). The auctioneer must answer bidder i immediately with q_i and P_i



Applications: CPU time, cache space, bandwidth…

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Incentive

A strategy (bid) $b_i(q)$ of player i is called ***dominant*** if for any other bid $\tilde{b}_i(q)$ and for any sequence of the past and future bids of the other players, $U_i(q_i, P_i) \geq U_i(\tilde{q}_i, \tilde{P}_i)$.

An auction is called ***incentive compatible*** if for any valuation $v_i(\cdot)$, declaring the true valuation is a dominant strategy.

Comments: the motivation of incentive – to free the bidders from strategic considerations (Vickrey et al. 1961); when all bidders are telling the truth, it is easy to maximize the social efficiency.

Supply Curves for On-line Auctions

Definition 1 (Supply curves). An on-line auction is called "based on supply curves" if before receiving the i'th bid it fixes a function (supply curve) $p_i(q)$ based on previous bids, and,

1. The quantity q_i sold to bidder i is the quantity q that maximizes the sum $\sum_{j=1}^{q} (b_i(j)-p_i(j))$, i.e. the bidder's utility.
2. The price paid by bidder i is $\sum_{j=1}^{q_i} p_i(j)$.

Assume each supply curve $p_i(q)$ is non-decreasing.

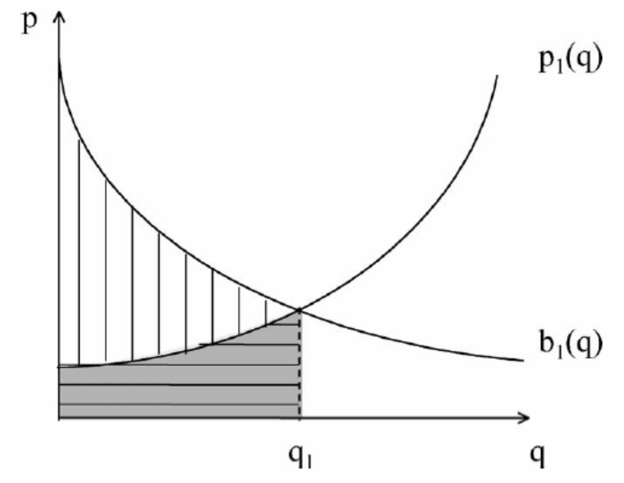


Fig. 1. An example of supply curves based auction.

Incentive & Supply Curves

Theorem 1. *A deterministic on-line auction is incentive compatible if and only if it is based on supply curves.*

Proof. This is proved in two directions by Lemma 1 and Lemma 2.

Lemma 1. An on-line auction that is based on supply curves is incentive compatible.

Proof. According to Definition 1, $\sum_{j=1}^{q}(b_i(j) - p_i(j))$ is maximized if based on supply curve. So that $U_i = \sum_{j=1}^{q}(v_i(j) - p_i(j))$ is always maximized iff $b_i(\cdot) = v_i(\cdot)$.

Lemma 2. Any deterministic incentive compatible on-line auction is based on supply curves.

Proof. Next slides.

Proof of Lemma 2

Lemma 2. Any deterministic incentive compatible on-line auction is based on supply curves.

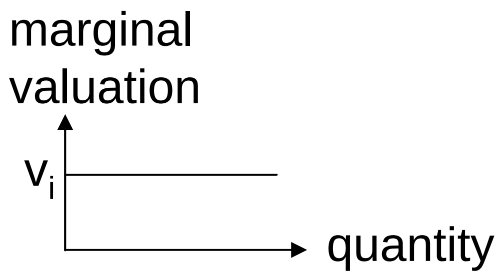
Proof.
For each player i, P_i is uniquely determined by q_i. Otherwise there exists bids v and v', where P<P', so that a player which has valuation v' will lie by declaring v to increase his utility, which contradicts incentive compatibility. Denote $P_i(q)$: $[1,k] \rightarrow R$, the total payment of player i for q items.

The allocation to player i must maximize $U_i(q) = \sum_{j=1}^{q} v_i(j) - P_i(q)$ otherwise player i will lie to increase his utility.

Denote $p_i(q) = P_i(q) - P_i(q-1)$. Since $P_i(0) = 0$, the allocation maximizes $U_i(q) = \sum_{j=1}^{q} v_i(j) - P_i(q) = \sum_{j=1}^{q} (v_i(j) - p_i(j))$, and the payment is $P_i(q_i) = \sum_{j=1}^{q_i} p_i(j)$ so that $p_i(q)$ is the supply curve according to Definition 1.

Special Case: Fixed Marginal Valuation

Lemma 3. Assume that for any player i, the marginal valuation is fixed to v_i. Then any incentive compatible on-line auction is based on non-decreasing supply curves.



Proof.

1. $q_i(v)$ is non-decreasing.

2. Define $p_i(q) = \inf \{ v \mid q_i(v) \geq q \}$. Since $q_i(v)$ is non-dreasing, $p_i(q)$ is non-decreasing as well.

3. Any incentive compatible on-line auction A is based on $p_i(q)$.

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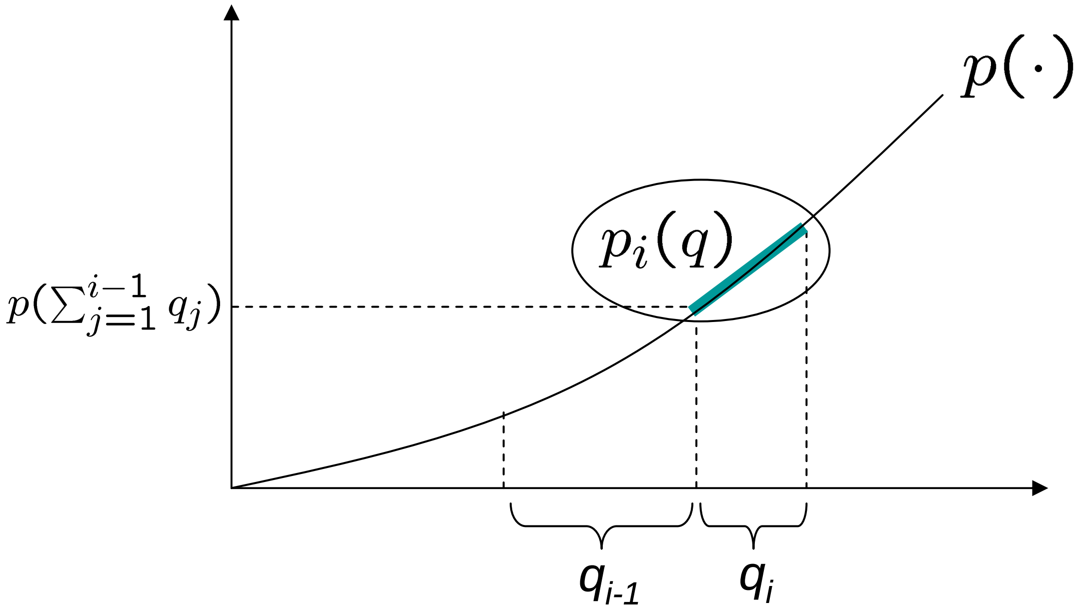
Global Supply Curve

Definition 2 (A global supply curve). An on-line auction is called "based on a global supply curve *p(q)*" if it is based on supply curves and if $p_i(q) = p(q + \sum_{j=1}^{i-1} q_j)$ where q_j is the quantity sold to the *j*th bidder.



Revenue and Social Efficiency

Definition 3 (Revenue and social efficiency). In auction A, for a valuation sequence σ, the *revenue* is

$$R_A(\sigma) = \sum_i P_i + \underline{p}(k - \sum_i q_i).$$

The *social efficiency* is

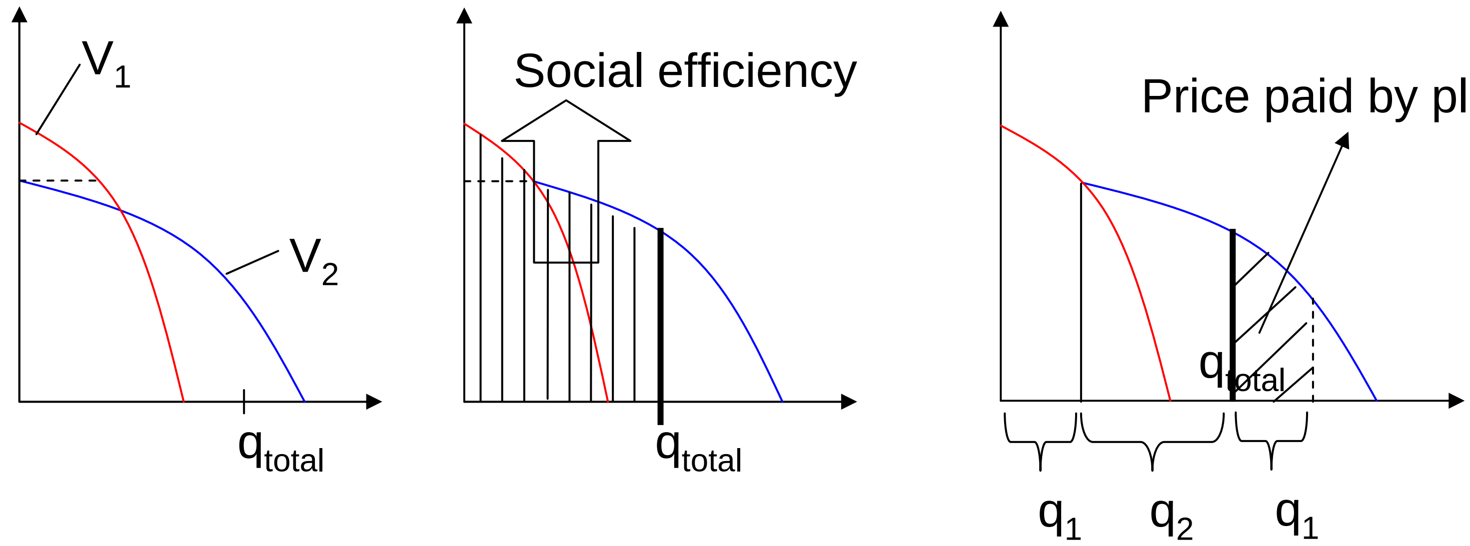
$$E_A(\sigma) = \sum_i \sum_{j=1}^{q_i} v_i(j) + \underline{p}(k - \sum_i q_i).$$

Assumptions:

1. All marginal valuations are taken from some known interval $[\underline{p}, \overline{p}]$, without assuming any distribution on them.
2. $\underline{p}$ is the salvage price of the auctioneer.

Off-line Vickrey Auction

Definition 4 (The Vickrey auction). In the Vickrey auction, each player declares his marginal valuation function. The allocation chosen is the one that maximizes the social efficiency (according to the players' declarations). The price charged from player i for the quantity q_i he receives is the worth of this additional quantity to the other players. Formally, denote by E_{-i} the optimal social efficiency when player i is missing, and by E the actual optimal social efficiency. Then the price that i pays is $E_{-i}-(E-v_i(q_i))$.



Competitiveness

Definition 5 (Competitiveness). An on-line auction A is *c-competitive with respect to the revenue* if for every valuation sequence σ, $R_A(\sigma) \geq R_{vic}(\sigma)/c$. Similarly, A is *c-competitive with respect to the social* efficiency if for every valuation sequence σ, $E_A(\sigma) \geq E_{vic}(\sigma)/c$.

Comments: E_{vic} is always optimal; while R_{vic} is not necessarily optimal, i.e. sometimes can be far from the optimal revenue.

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One Divisible Good

Let *c* be the unique solution to the equation:

$$c = ln\frac{(\overline{p}/\underline{p}) - 1}{c - 1} \qquad (1)$$

Comments: it can be shown that $c = \Theta(ln(\overline{p}/\underline{p}))$. For example, if $(\overline{p}/\underline{p})$=2 then c=1.28, and if $(\overline{p}/\underline{p})$=8 then c=1.97.

Definition 6 (The competitive on-line auction). Define the *competitive supply curve* by

$$p(x) = \underline{p}(1 + (c - 1)e^{cx}).$$

The *competitive on-line auction* has the competitive supply curve as its global supply curve.

One Divisible Good

Let $q(x) = p^{-1}(x)$ and $r(x) = \int_0^{q(x)} p(y)dy$

Lemma 4. (El-Yaniv et al) The functions *q(x), r(x)* preserve the following conditions:

1. $\forall x \leq c \cdot \underline{p} : q(x) = 0, r(x) = 0$
2. $\forall x > c \cdot \underline{p} : r(x) + \underline{p} \cdot (1 - q(x)) = x/c$
3. $q(\overline{p}) = 1$

Where *c* is as defined in *Eq.* (1).

One Divisible Good

Theorem 2. *The competitive on-line auction is c-competitive with respect to the revenue and the social efficiency.*

Lemma 6. For any sequence of valuations σ, $R_{cola}(\sigma) \geq R_{vic}(\sigma)/c$, where "cola" is the competitive on-line auction and "vic" is the Vickrey auction.

Lemma 7. For any sequence of valuations σ, $E_{cola}(\sigma) \geq E_{opt}(\sigma)/c$, where $E_{opt}(\sigma)$ is the optimal social efficiency for σ.

One Divisible Good

Theorem 3. *Any incentive compatible on-line auction must have a competitive ratio of at least c with respect to both the revenue and the social efficiency, where c is the solution to Eq. (1).*

Lemma 5. For any constant $\tilde{c} < c$, there is no function $\tilde{q}(x)$ such that

$$\forall x \in [\underline{p}, \overline{p}], \tilde{r}(x) + \underline{p} \cdot (1 - \tilde{q}(x)) \geq x/\tilde{c},$$

Where $\tilde{r}(x) = \int_0^{\tilde{q}(x)} \tilde{p}(t)dt$ and $\tilde{p}(x) = \tilde{q}^{-1}(x)$

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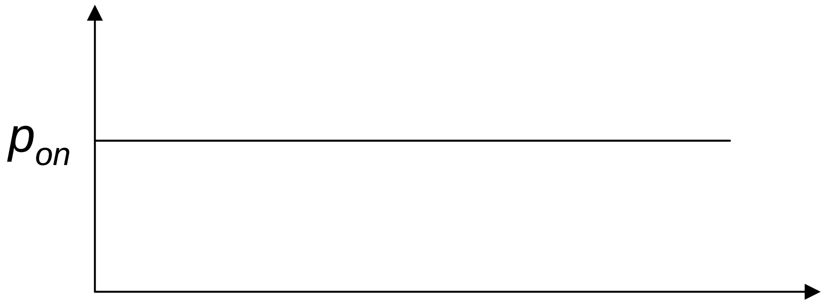
A Randomized Auction for *k* Indivisible Goods

Definition 7. The randomized on-line auction: the supply curve is fixed with $p(x)=p_{on}$, where p_{on} is chosen by using the cumulative distribution q(·).



Theorem 4. *For any sequence of valuations σ, the expected revenue of the randomized auction is at least 1/c times the optimal efficiency, i.e. $E(R_{on}(\sigma)) \geq E_{opt}(\sigma)/c$.*

Proof of Theorem 4

Define the cdf: $\forall v \in [\underline{p}, \overline{p}]$, $Pr(x \leq v) = q(v)$, $f(x) = \mathrm{d}[q(x)]/\mathrm{d}x$

$$E(R_{\text{on}} | v_{i+1} \leqslant p_{\text{on}} \leqslant v_i) \geqslant \int_{v_{i+1}}^{v_i} i \cdot x \cdot \frac{f(x)}{Pr(v_{i+1} \leqslant p_{\text{on}} \leqslant v_i)} \mathrm{d}x + \underline{p}(k-i)$$

$$E(R_{\text{on}}) \geqslant \sum_{i=0}^{k} \left[i \cdot \int_{v_{i+1}}^{v_i} x f(x) \, \mathrm{d}x + \underline{p}(k-i) \cdot Pr(v_{i+1} \leqslant p_{\text{on}} \leqslant v_i) \right]$$

$$= \sum_{i=0}^{k} \left[i \cdot \int_{v_{i+1}}^{v_i} x f(x) \, \mathrm{d}x + \underline{p}(k-i)(q(v_i) - q(v_{i+1})) \right].$$

$$\geqslant \sum_{i=1}^{k} \left[\int_{\underline{p}}^{v_i} x f(x) \, \mathrm{d}x \right] + \sum_{i=1}^{k} [\underline{p}(q(v_0) - q(v_i))]$$

$$= \sum_{i=1}^{k} \left[\int_{\underline{p}}^{v_i} x f(x) \, \mathrm{d}x + \underline{p}(1 - q(v_i)) \right] = \sum_{i=1}^{k} \frac{v_i}{c} = E_{\text{opt}}(\sigma)/c$$

A Deterministic Auction for *k* Indivisible Goods

Definition 8 (The discrete on-line auction). The discrete on-line auction is based on the following global supply curve:

$$p(j) = \underline{p} \cdot \Phi^{j/(k+1)}, \text{ for } j = 1, \cdots, k.$$

$$\text{assume w.l.o.g that } \underline{p} = 1, \bar{p} = \phi.$$

Theorem 5. *The discrete on-line auction is $k \cdot \Phi^{1/(k+1)}$-competitive with respect to the revenue and to the social efficiency. When $k \geq 2 \cdot \ln \Phi$ then the discrete on-line auction is also $2 \cdot e \cdot (\ln(\Phi)+1)$-competitive with respect to the revenue and to the social efficiency.*

Theorem 6. *Any incentive compatible on-line auction of k goods has a competitive ratio of at least $m = \max\{\Phi^{1/(k+1)}, c\}$ with respect to the revenue and to the efficiency.*

Revenue Analysis on Uniform Distribution

We compare the expected revenue of the competitive on-line auction to the expected revenue of the Vickrey off-line auction for a divisible good in the special case of fixed marginal valuations uniformly distributed in $[\underline{p}, \overline{p}]$.

	On-line revenue	Vickrey revenue
$\bar{p}=1.5,\ n=2$	1.15	1.17
$\bar{p}=3,\ n=2$	1.60	1.67
$\bar{p}=10,\ n=2$	3.33	4.00
$\bar{p}=2,\ n=2$	1.31	1.33
$\bar{p}=2,\ n=3$	1.37	1.50
$\bar{p}=2,\ n=100$	1.56	1.98