

On a Network Creation Game

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PODC '03, pages 347-351

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Network

- Internet, social network
 - cell, universe, brain, ...
- Internet's open architecture
 - autonomous (“selfish”)
 - *hardware costs*
 - *communication costs*

Question

What is the *price* of the Internet's open architecture?
— the (worst-case) price of anarchy

Network Creation Game

Create undirected graph by selfish nodes without coordination

- Each node lays down edges to a subset of other nodes
- Once edge is laid down, anyone can use it
- The graph is built from the union of these sets of edges
- Cost of each node
 - *hardware cost*: the total cost of the edges laid down by this node
 - *communication cost*: the sum of the distances from the node to all others
- Social cost: the sum of all nodes' costs

Formal Definition

- n players: $[n] = \{1, 2, \dots, n\}$
- Strategy space of player i : $S_i = 2^{[n]-\{i\}}$
- A play: $s = (s_1, \dots, s_n) \in S_1 \times \dots \times S_n$
- The cost of player i under play s is

$$c_i(s) = \alpha \cdot |s_i| + \sum_{j=1}^n d_{G[s]}(i, j).$$

- $G[s] = (V, E)$ is the resulting undirected graph
— $V = [n]$, $E = \bigcup_{i=1}^n (\{i\} \times s_i)$
- $d_{G[s]}(i, j)$ is the (shortest path) distance between player i and j in the graph $G[s]$, which is infinity if there are no path between i and j .

Nash Equilibrium

- A (pure) Nash equilibrium in this game is a play $s = (s_1, \dots, s_n)$, such that for each player i , and for all $s' = (s_1, \dots, s'_i, \dots, s_n)$, $c_i(s) \leq c_i(s')$

Observation

At any Nash equilibrium s , the graph $G[s]$ is connected, and the strategies s_i are mutually disjoint.

Social Cost

- Social cost when the strategies s_i are mutually disjoint:

$$\begin{aligned}C(G) &= \sum_i c_i = \sum_i (\alpha \cdot |s_i| + \sum_{j=1}^n d_G(i, j)) \\ &= \alpha |E| + \sum_{i,j} d_G(i, j),\end{aligned}$$

satisfied by all Nash equilibria and social optima.

- A trivial lower bound:

$$\begin{aligned}C(G) &\geq \alpha |E| + 2 |E| + 2(n(n-1) - 2 |E|) \\ &= 2n(n-1) + (\alpha - 2) |E|,\end{aligned}$$

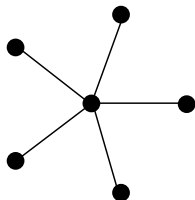
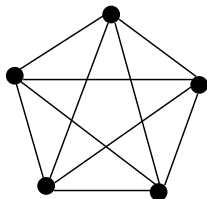
the equality is achieved by any graph of diameter at most 2.

Basic Results: Social Optimum

$$C(G) \geq 2n(n-1) + (\alpha - 2)|E|$$

- $\alpha < 2$
 - The social optimum is the complete graph — maximize $|E|$

- $\alpha \geq 2$
 - The social optimum is the star — minimize $|E|$



Basic Results: Nash Equilibrium

$$c_i(s) = \alpha \cdot |s_i| + \sum_{j=1}^n d_{G[s]}(i, j)$$

Any Nash equilibrium cannot miss any edge whose addition would reduce the second term by more than α

- $\alpha < 1$
 - The complete graph is the only Nash equilibrium — of diameter 1
- $1 \leq \alpha < 2$
 - The worst Nash equilibrium is the star — of diameter at most 2, $|E|$ minimized
 - The price of anarchy is

$$\begin{aligned}\rho(\text{star}) &= \frac{C(\text{star})}{C(K_n)} = \frac{(n-1)(\alpha-2+2n)}{n(n-1)(\frac{\alpha-2}{2}+2)} \\ &= \frac{4}{2+\alpha} - \frac{4-2\alpha}{n(2+\alpha)} < \frac{4}{2+\alpha} \leq \frac{4}{3}\end{aligned}$$

- $\alpha \geq 2$
 - The star is a Nash equilibrium, but worse one may exist

Upper Bounds

- $\alpha \geq n^2$
 - Nash equilibrium must be a tree
 - Recall that the social optimum is the star
 - The price of anarchy is $O(1)$

Theorem

For any tree Nash equilibrium T , $\rho(T) < 5$.

Constant Upper Bound on Tree Nash Equilibrium

Fact: Every tree has a separator (a node that, when removed, yields no components with more than $n/2$ nodes).

Proof: Assume $\alpha \geq 2$. Let n be the number of nodes in T . Find a separator $z \in T$. Let $d \geq 2$ be the depth of T when rooted at z .

For some leaf u at depth d , a link from u to z could save at least $(d-1)(n/2)$ to u . Thus, $\alpha \geq n(d-1)/2 > (d-1)(n/2-1)$. Since $\text{diam}(T) \leq 2d$, so $\text{diam}(T) \leq 2 + 4\alpha/(n-2)$. Then,

$$\begin{aligned} C(T) &\leq \alpha(n-1) + \left(2 + \frac{4\alpha}{n-2}\right)(n-2)(n-1) + 2(n-1) \\ &= 5\alpha(n-1) + 2(n-1)^2. \end{aligned}$$

So we have,

$$\rho(T) = \frac{C(T)}{C(\text{star})} \leq \frac{5\alpha(n-1) + 2(n-1)^2}{\alpha(n-1) + 2(n-1)^2} < 5.$$

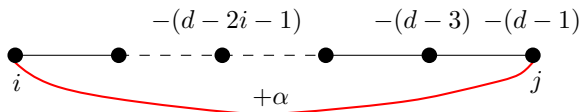
Upper Bound: $\alpha < n^2$

Theorem

If $\alpha < n^2$, the price of anarchy is $O(\sqrt{\alpha})$.

Lemma

If G is a Nash equilibrium, $d_G(i, j) \leq 2\sqrt{\alpha}$ for every i and j .



Proof: It follows from

$$+\alpha - \sum_{i=0}^{\lfloor \frac{d-1}{2} \rfloor} [(d-i) - (i+1)] \geq 0.$$

Proof

- A counting argument can show that $|E| = O(\frac{n^2}{\sqrt{\alpha}})$.
- Therefore,

$$\begin{aligned}C(G) &\leq \alpha \cdot O\left(\frac{n^2}{\sqrt{\alpha}}\right) + 2\sqrt{\alpha} \cdot n(n-1) \\ &\leq O(\sqrt{\alpha}n^2)\end{aligned}$$

- Since

$$C(\text{star}) = \alpha(n-1) + 2(n-1)^2 = \Omega(n^2),$$

the price of anarchy is

$$\rho(G) = \frac{C(G)}{C(\text{star})} = O(\sqrt{\alpha}).$$

Counting argument: $|E| = O\left(\frac{n^2}{\sqrt{\alpha}}\right)$

Let $T_i = \{u \in V : \text{the shortest path from } v \text{ to } u \text{ goes through } e_i\}$,
where $e_i = (v, w) \in E$, and e_i is laid down by v .

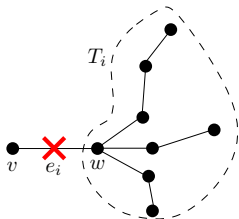
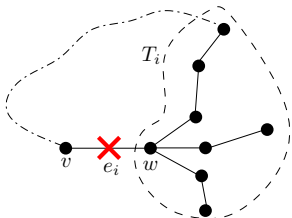
- T_i and v are connected in $G' = (V, E - e_i)$,
then for every $u \in T_i$,

$$d_{G'}(v, u) - d_G(v, u) < 2 \cdot \text{diam}(G) \leq 4\sqrt{\alpha}.$$

Since $+\alpha - \sum_{u \in T_i} (d_{G'}(v, u) - d_G(v, u)) \leq 0$,
 $|T_i| = \Omega(\sqrt{\alpha})$.

— associate $|T_i| - 1 = \Omega(\sqrt{\alpha})$ non-edges
with e_i .

- T_i and v are not connected in G'
— associate $|T_i| - 1 + |V - T_i| - 1$
 $= n - 2 = \Omega(\sqrt{\alpha})$ non-edges with e_i .



A Lower Bound

Theorem

For any $\varepsilon > 0$, there exists a Nash equilibrium of the network creation game with the price of anarchy greater than $3 - \varepsilon$.

- $T_{k,d}$: an outward-directed (which means that edges are laid down by parent nodes) complete k -ary tree of depth d , where $k \geq 3$ and $d \geq 2$. Set $\alpha = (d - 1)n$.
- $T_{k,d}$ is a Nash equilibrium of the network creation game.
- By counting distances between leaves alone, we have

$$C(T_{k,d}) \geq \alpha(n - 1) + 2d \cdot \left(\frac{(k - 1)n}{k^2}\right)^2 \cdot k(k - 1)$$

- Since $C(\text{star}) = \alpha(n - 1) + 2(n - 1)^2$, we have

$$\lim_{k,d \rightarrow \infty} \rho(T_{k,d}) = \lim_{d \rightarrow \infty} \frac{(d - 1)n(n - 1) + 2dn^2}{(d - 1)n(n - 1) + 2(n - 1)^2} = 3.$$

Recent Improvements

On Nash Equilibria for a Network Creation Game

Susanne Albers, Stefan Eilts, Eyal Even-Dar, Yishay Mansour
and Liam Roditty

SODA '06, pages 89-98

- $\alpha \geq 12n \log n$
 - Every Nash equilibrium is a tree
 - the price of anarchy is ≤ 1.5
- $\alpha < 12n \log n$
 - the price of anarchy is bounded by

$$15(1 + (\min\{\frac{\alpha^2}{n}, \frac{n^2}{\alpha}\})^{1/3})$$

- constant if $\alpha \in O(\sqrt{n})$
- a worst case bound of $O(n^{1/3})$ (at $\alpha = n$, instead of $O(n)$)

Thank You!