Addendum

This page proves the statement

\[ \text{if } y \in P_i \text{ with } y \leq t \text{ then } \exists z \in L_i \text{ such that } \]

\[ \left( 1 - \frac{\epsilon}{n} \right)^i y \leq z \leq y \]  \hspace{1cm} (1)

that appeared in the analysis of the subset-sum approximation scheme presented in COMP572. Please see the class notes for definitions of the terms.

The proof will be by induction on \( i \). When \( i = 1 \) note that \( P_1 = < 0, x_1 > \) while \( L_1 = < 0, x_1 > \) or \( L_1 = < 0 > \), depending upon whether or not \( x_i \leq t \). In both cases, if \( y \in P_1 \) with \( y \leq t \) then \( y \in L_1 \) so (1) is true.

Now assume that (1) is true for \( i \). We will prove its correctness for \( i + 1 \). Recall that \( P_{i+1} = \text{Merge}(P_i, P_i + x_{i+1}) \) with all items \( > t \) thrown out and \( L_{i+1} \) is the trimmed version of \( \text{Merge}(L_i, L_i + x_{i+1}) \) with all items \( > t \) thrown out.

So now suppose that \( y \in P_{i+1} \) with \( y \leq t \).

There are two cases: (i) \( y \in P_i \) or (ii) \( y \in P_i + x_{i+1} \).

If \( y \in P_i \) then, by induction, \( \exists z_i \in L_i \) such that \( (1 - \frac{\epsilon}{n})^i y \leq z_i \leq y \). Since \( z_i \in \text{Merge}(L_i, L_i + x_{i+1}) \), \( \exists z \in L_{i+1} \) such that \( (1 - \frac{\epsilon}{n}) z_i \leq z \leq z_i \). Combining the two sets of inequalities yields

\[ \left( 1 - \frac{\epsilon}{n} \right)^{i+1} y \leq \left( 1 - \frac{\epsilon}{n} \right) z_i \leq z \leq z_i \leq y \]

which is what we wanted to show.

If \( y \in P_i + x_{i+1} \) then \( y = y_i + x_{i+1} \) for some \( y_i \in P_i \). Again by induction \( \exists z_i \in L_i \) such that \( (1 - \frac{\epsilon}{n})^i y_i \leq z_i \leq y_i \). Therefore

\[ \left( 1 - \frac{\epsilon}{n} \right)^i y = \left( 1 - \frac{\epsilon}{n} \right)^i (y_i + x_{i+1}) \leq \left( 1 - \frac{\epsilon}{n} \right)^i y_i + x_{i+1} \leq z_i + x_{i+1} \leq y_i + x_{i+1} = y. \]

Since \( z_i + x_{i+1} \in \text{Merge}(L_i, L_i + x_{i+1}) \), \( \exists z \in L_{i+1} \) such that \( (1 - \frac{\epsilon}{n}) (z_i + x_{i+1}) \leq z \leq z_i + x_{i+1} \). Combining the two sets of inequalities yields

\[ \left( 1 - \frac{\epsilon}{n} \right)^{i+1} y \leq \left( 1 - \frac{\epsilon}{n} \right) (z_i + x_{i+1}) \leq z \leq z_i + x_{i+1} \leq y \]

which is what we wanted to show and the proof is now complete.