# Fibonacci Heaps

CLRS: Chapter 20

*Last Revision: July 28, 2007*

| Procedure | Binary heap (worst-case) | Binomial heap (worst-case) | Fibonacci heap (amortized) |
|---|---|---|---|
| Make-Heap | $\Theta(1)$ | $\Theta(1)$ | $\Theta(1)$ |
| Insert | $\Theta(\lg n)$ | $O(\lg n)$ | $\Theta(1)$ |
| Minimum | $\Theta(1)$ | $O(\lg n)$ | $\Theta(1)$ |
| Extract-Min | $\Theta(\lg n)$ | $\Theta(\lg n)$ | $O(\lg n)$ |
| Union | $\Theta(n)$ | $O(\lg n)$ | $\Theta(1)$ |
| Decrease-Key | $\Theta(\lg n)$ | $\Theta(\lg n)$ | $\Theta(1)$ |
| Delete | $\Theta(\lg n)$ | $\Theta(\lg n)$ | $O(\lg n)$ |

So far we have seen*Binomial heaps* and learnt some techniques for performing *amortized analysis*.

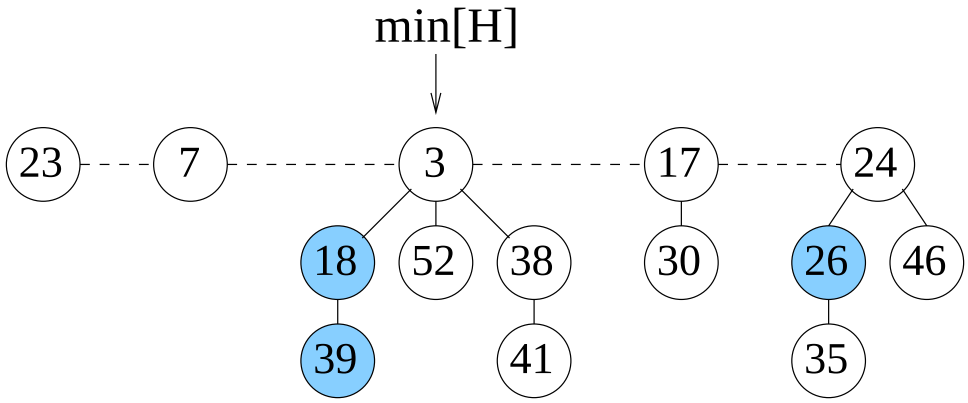
In this section we will design *Fibonacci heaps*, whose running times will be *amortized* and not worst case. Even with only amortized running times, Fibonacci heaps provide enough of an improvement to reduce the run-time of Dijkstra's and Prim's algorithms from $(|V| + |E|) \log |V|$ down to $|V| \log |V| + |E|$.

Our amortized analysis will use the Potential method.

*Fibonacci heaps*, like *Binomial heaps*, are a *collection* of heap-ordered trees.

Some properties

- nodes in a F.H are **not** ordered (by degree) in the root list or as siblings.
- (root and sibling) lists kept as *circularly-linked* lists. Allows constant time deletion/insertion/concatenation.
- Each node stores its degree (number of children).
- $min[H]$ is a pointer to minimum root in root list.
- $N(H)$ keeps *number* of nodes currently in $H$.




**Marked Nodes:**

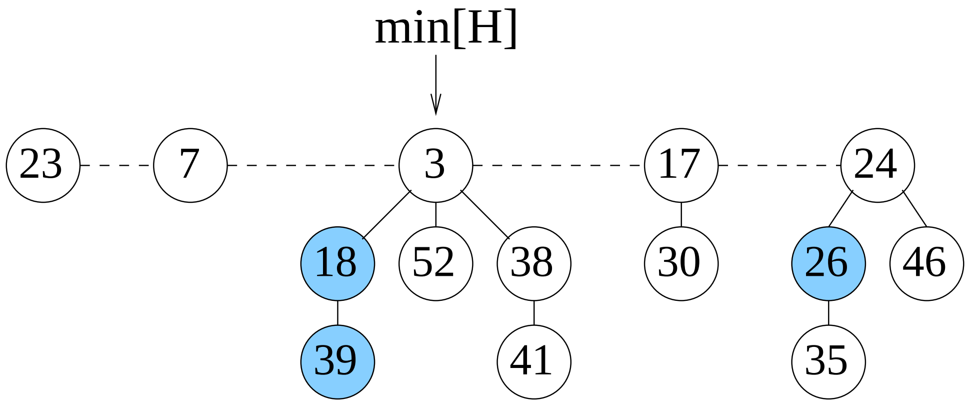
Some nodes will be **marked**
(indicated by the *marked* bit set to 1).
(i) A node $x$ will be marked if $x$ has lost a child since the last time that $x$ was made a child of another node.
(ii) Newly created nodes are unmarked
(iii) When node $x$ becomes child of another node it becomes unmarked.

**Potential Function:**

The *potential* of $H$ will be $t(H)$, the number of nodes in root list of $H$ plus two times $m(H)$, the number of marked nodes.

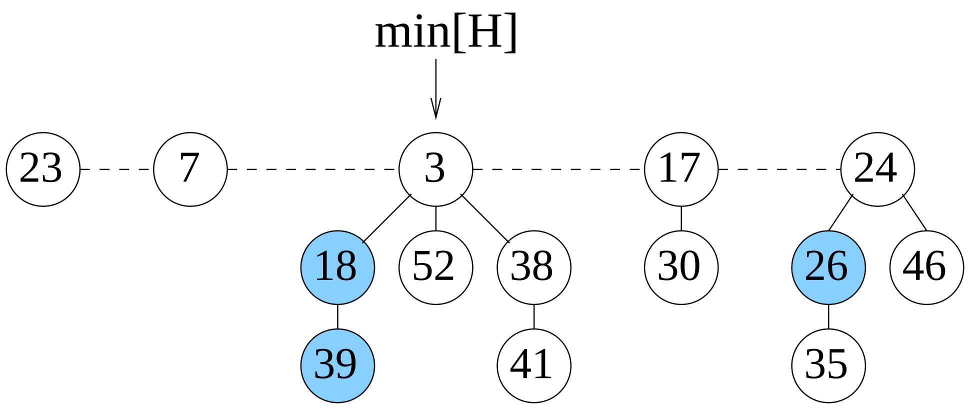
$$\Phi(H) = t(H) + 2m(H).$$

In example above $\Phi(H) = 5 + 2 \cdot 3 = 11$.



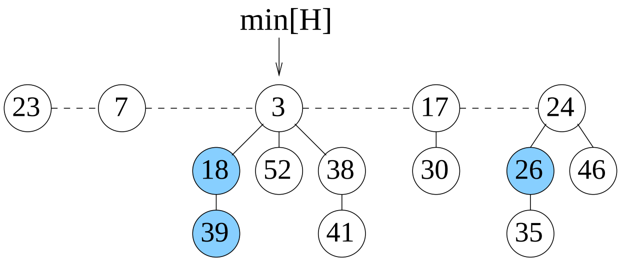

**Assumption:** There is a maximum degree $D(n)$ on the degree of any node in an $n$-node Fibonacci heap.

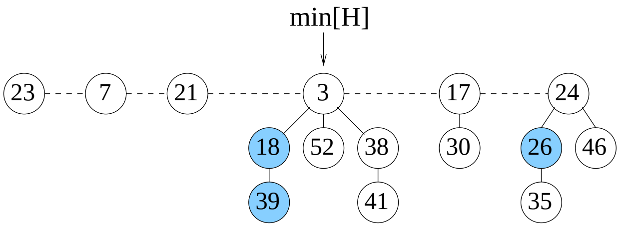
We will prove later that $D(n) = O(\log n)$.




**Make-Heap():**
This is a very easy $O(1)$ (both amortized and actual) operation.

**Minimum($H$):** Return the node pointed to by $min[H]$. This takes $O(1)$ actual time.
The heap does not change before and after this operation so difference in potential is $0$.
Amortized cost is then also $O(1)$.






## H'=Insert($H, x$):

Create new tree containing $x$ & add it to root list.
$Min[H] = \mathrm{min}(Min[H], x)$.
Update pointers appropriately

**Do not** combine items in the root list.
Clean up will be done during Extract-Min($H$).
If $k$ nodes inserted into $H$, then $H$ becomes a linked list with $k$ single nodes.

Actual cost $c$ of operation is $O(1)$.
$t(H') = t(H) + 1$; $m(H') = m(H)$ so
$\Phi(H') - \Phi(H) = ((t(H') + 2m(H')) - (t(H) + 2m(H))) = 1$
and amortized cost satisfies
$\hat{c} = c + 1 = O(1)$.

**H=Union($H_1, H_2$):**
Just concatenate the two root lists of $H_1$ and $H_2$.
**Do not** combine items in the root list.
Set $min[H] = \mathrm{min}(min[H_1], min[H_2])$.

Actual cost of this operation is $c = O(1)$.

Concatenating the root lists does not change the total number of items in the root lists or the total number of marked nodes so change in potential is

$$\begin{aligned}\Phi(H) &- (\Phi(H_1) + \Phi(H_2)) \\ &= (t(H) + 2m(H)) - \\ &\quad ((t(H_1) + 2m(H_1)) + (t(H_2) + 2m(H_2)) \\ &= 0\end{aligned}$$

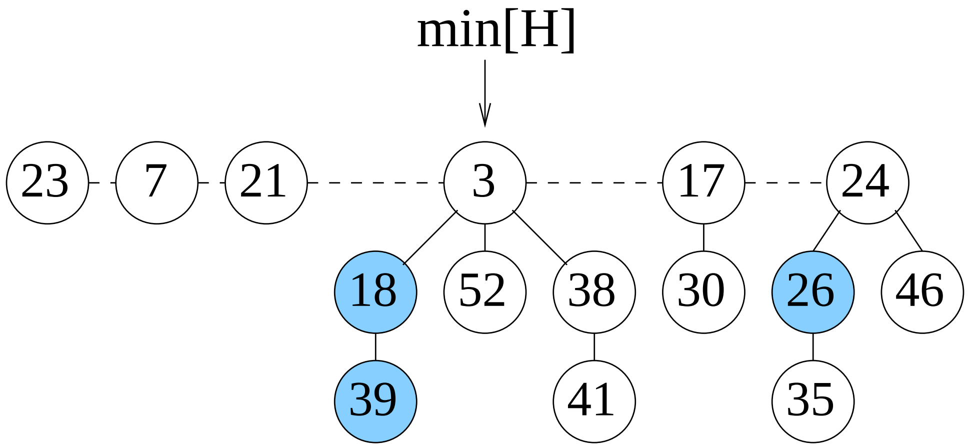
and amortized cost is

$\hat{c} = c + 0 = O(1)$.

**Extract-Min($H$):**
This is the most complicated operation.
It is here where we *clean-up* large root lists.
At the end of this operation, root list will contain
at most one root of each possible degree.
This implies that root list contains
$\le D(n-1)+1$ nodes.

**Extract-Min($H$)** is quite similar to same operation in binomial heaps.
Let $A$ be tree with root $min[H]$.
Extract $A$ from $H$.
Remove the root of $A$;
reinsert remaining trees back into root list of heap.
update $min[H]$ during this procedure.
Link roots of equal degree until at most one root remains of each degree.

Let $x$ be root of tree $X$, $y$ root of $Y$.
Assume w.l.o.g. that $key[x] \le key[y]$.
When linking $X$ and $Y$ point $y$ to $x$ and
increment degree($x$) and set mark($y$)=0.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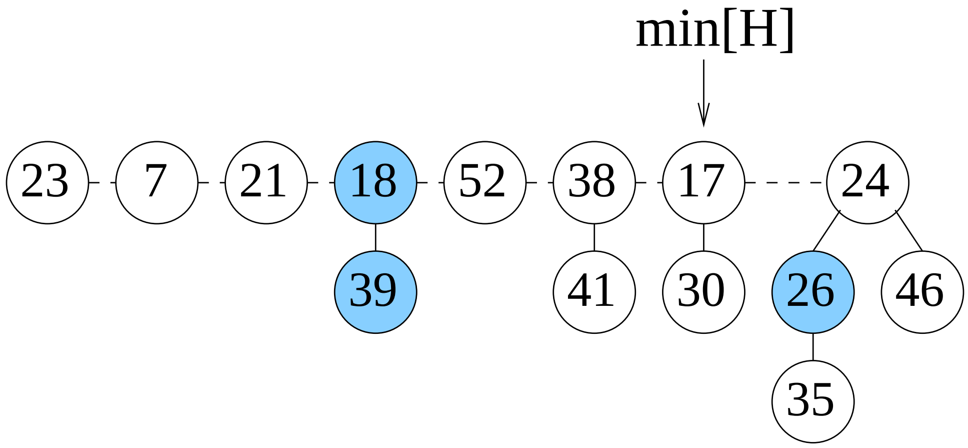
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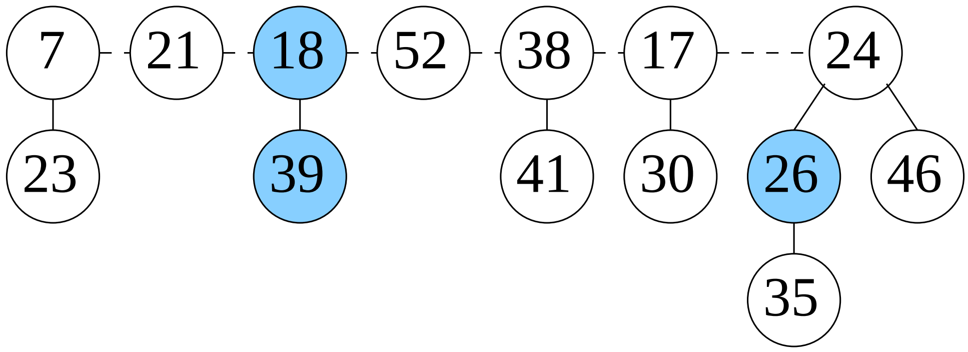
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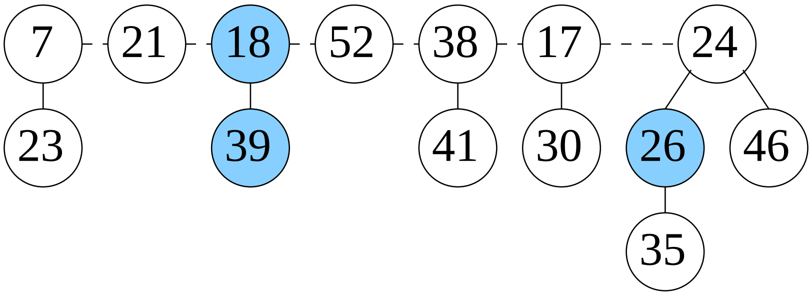
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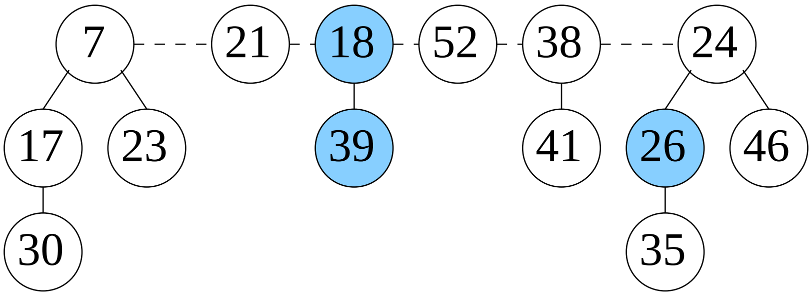
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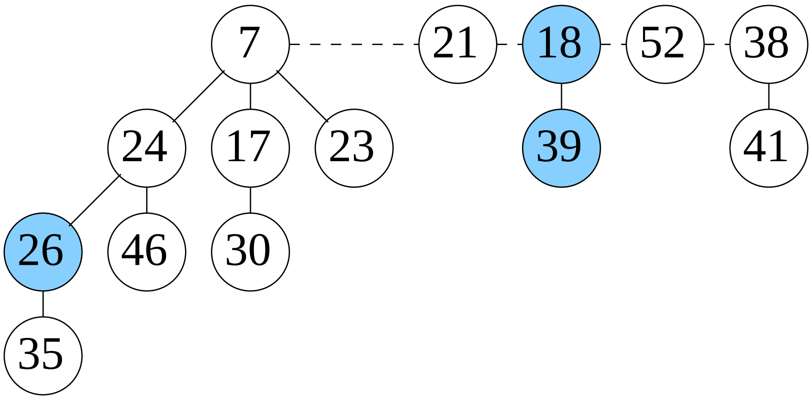
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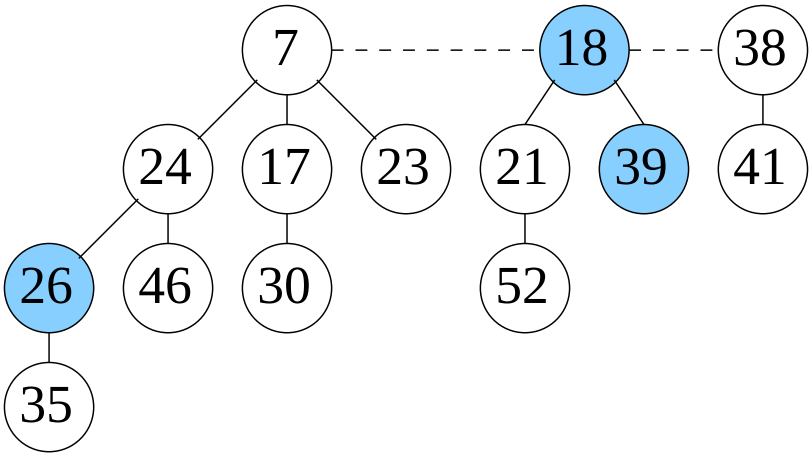
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**Actual Cost:**

$A$ has at most $D(n)$ children.

After concatenation there will be at most

$t(H) + D(n) - 1$ nodes on root list.

Linking any two trees requires $O(1)$.

Total cost of concatinating, linking and updating $min[H]$ is $O(t(H) + D(n))$.

Actual cost is then $c = O(t(H) + D(n))$.

**Potential:**

Original potential is $\Phi(H) = t(H) + 2m(H)$.

Marked nodes can not be created by operation;

only cleared.

After all of the linking there will be at most $D(n-1)+1$ nodes on root list (Why?).

Final potential is then

$\Phi(H') = t(H') + 2m(H') \le D(n) + 1 + 2m(H)$.

and

$\Phi(H') - \Phi(H) \le D(n) + 1 + 2m(H) - (t(H) + 2m(H)) = D(n) + 1 - t(H)$.

**Actual Cost:**
$c = O(t(H) + D(n))$.

**Potential:**
$\Phi(H') - \Phi(H) \leq D(n) + 1 - t(H)$.

**Amortized Cost:**
If we scale the units of potential large enough then amortized cost is

$$\begin{aligned} \hat{c} &= c + \Phi(H') - \Phi(H) \\ &\leq O(t(H) + D(n)) + D(n) + 1 - t(H) \\ &= O(D(n)) \end{aligned}$$

so

$$\hat{c} = O(\log n).$$

**Decrease-Key($H, x, k$):**
Let $p[x] = parent[x]$.

This operation is *very* different from any we've seen before.

We actually Cut subtree rooted at $x$ out of the tree, move it to the root list and unmark $x$.

We then look at $p[x]$;
if it wasn't marked, we mark it and stop.

If it was marked, we cut $p[x]$, unmark it, move *it* to the root list, and then check $p[p[x]]$, cascading this process up until either an unmarked ancestor or the root is found.

Decrease-Key($H, x, k$)

(i) First check if $k < key[p[x]]$. If not, do nothing.

(ii) Otherwise

    Cut($H, x, p[x]$)
    CascadingCut($H, p[x]$)

(iii) If $k < key[min[H]]$ then

    $min[H] = x$

Cut($H, x, y$):

(i)Move tree rooted at $x$ to root list;

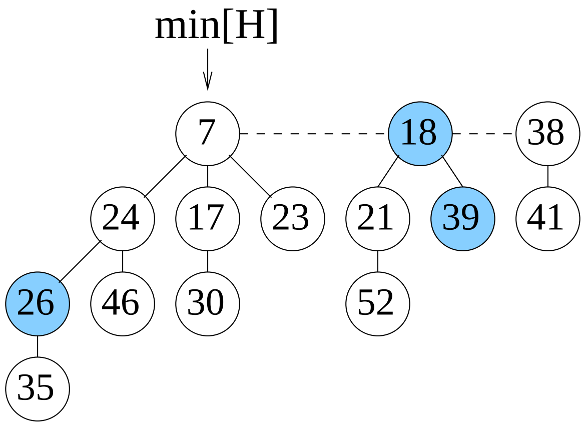
    decrement $degree[y]$.

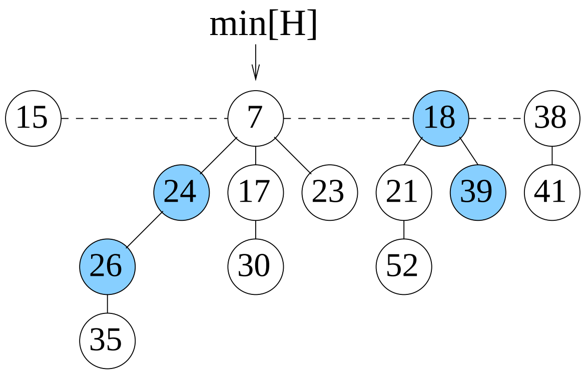
(ii) set $mark[x] =$ false

CascadingCut($H, y$)

**If** $y$ not in root list

    **then if** $mark[y] ==$ false
        **then** $mark[y] =$ true
        **else** Cut($H, y, p[y]$)
            CascadingCut($H, p[y]$)

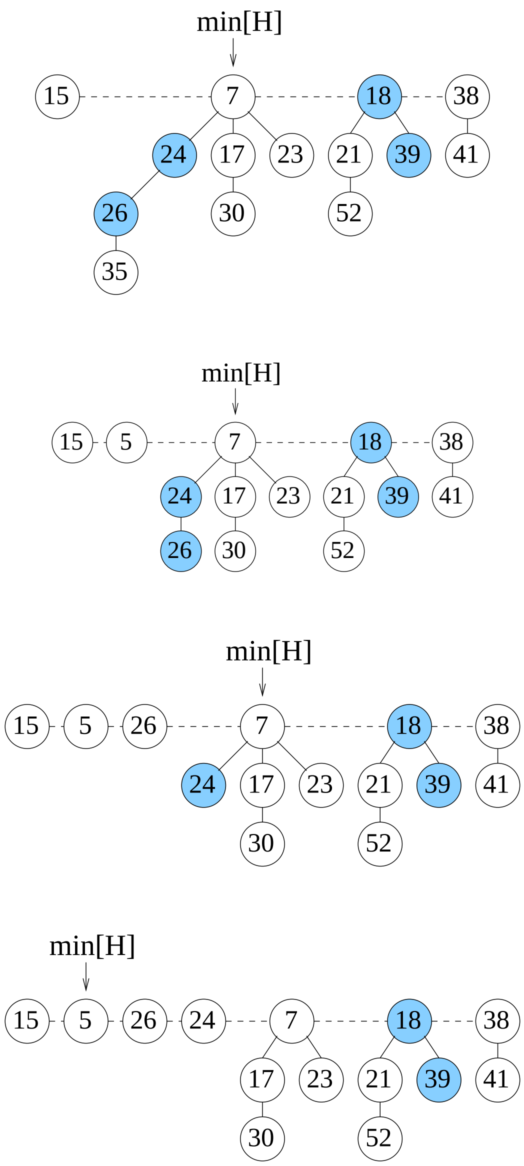



Node containing 46, has key decreased to 15.

It is cut and moved to root list.

Its parent, 24, is then marked.




35 changed to 5, cut & moved to root
Its parent 26 was marked; so it's cut as well,
moved to root and mark cleared.
26's parent 24 was marked; so it's cut as well,
moved to root and mark cleared.
24's parent 7 is in root list so cascade terminates.

## Decrease-Key($H, x, k$): Run Time

**Actual Cost:**
If CascadingCut is recursively called $d$ times,
$c = O(d+1)$.

**Potential:**
Original potential is $\Phi = t(H) + 2m(H)$.

After operation there are
$t(H) + d$ trees in root list and at most
$m(H) - (d-1) + 1 = m(H) - d + 2$
marked nodes. Then

$$\Phi(H') - \Phi(H) \leq 4 - d.$$

**Amortized Cost:**
If we scale the units of potential to be large enough then

$$\begin{aligned} \hat{c} &= c + \Phi(H') - \Phi(H) \\ &\leq O(d+1) + 4 - d \\ &= O(1) \end{aligned}$$

We can now begin to understand why the marked nodes contribute $2m(H)$ to potential.

The first unit of potential for each marked node was to pay for a step in the cascaded cut.

The second unit was to pay for the increase in potential caused by a cut node becoming a root, which in turn pays for a later linking of that root to another root during an Extract-Min.

**Delete($H, x$):**

This is equivalent to

an amortized $O(1)$ time **Decrease-Key($H, x, -\infty$)**

and

an amortized $O(\log n)$ time **Extract-Min($H$)**

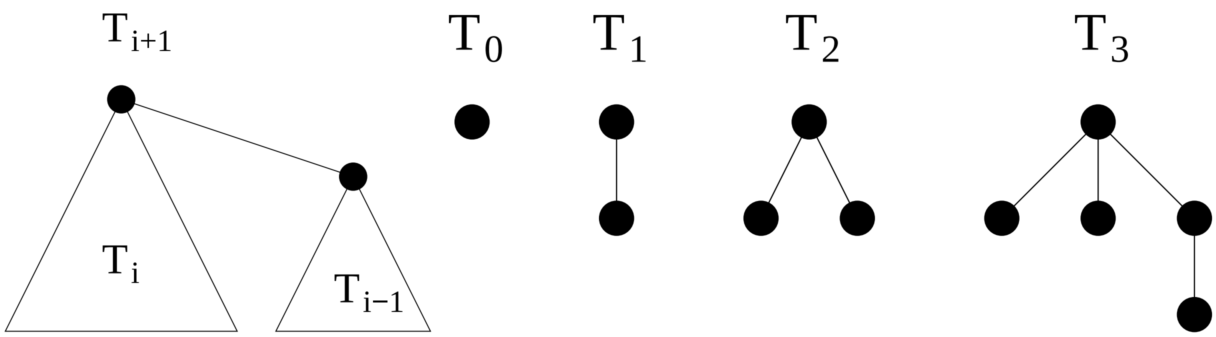
| Procedure | Binary heap (worst-case) | Binomial heap (worst-case) | Fibonacci heap (amortized) |
|---|---|---|---|
| Make-Heap | $\Theta(1)$ | $\Theta(1)$ | $\Theta(1)$ |
| Insert | $\Theta(\lg n)$ | $O(\lg n)$ | $\Theta(1)$ |
| Minimum | $\Theta(1)$ | $O(\lg n)$ | $\Theta(1)$ |
| Extract-Min | $\Theta(\lg n)$ | $\Theta(\lg n)$ | $O(\lg n)$ |
| Union | $\Theta(n)$ | $O(\lg n)$ | $\Theta(1)$ |
| Decrease-Key | $\Theta(\lg n)$ | $\Theta(\lg n)$ | $\Theta(1)$ |
| Delete | $\Theta(\lg n)$ | $\Theta(\lg n)$ | $O(\lg n)$ |

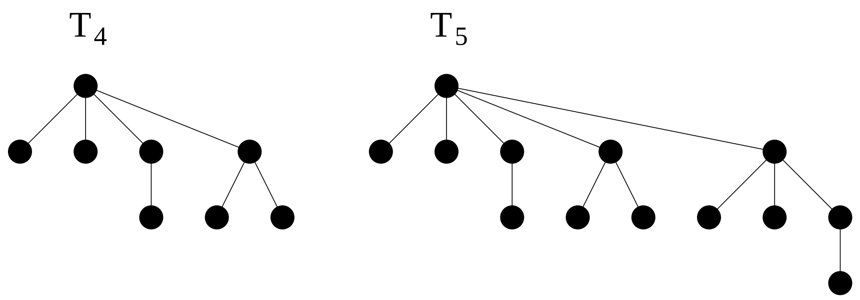
We have demonstrated that Fibonacci Heaps satisfy the stated amortized running times under the assumption that $D(n) = O(\log n)$ where $D(n)$ is the maximum degree of a tree in a Fibonacci Heap containing $n$ nodes.

We still have to

- Prove that $D(n) = O(\log n)$
- Explain why the data structure is called a *Fibonacci* heap.

Define trees $T_i$ as follows: $T_0$ is a single node, $T_1$ is a node with one child and, for $i > 1$, $T_i$ is constructed by pointing the root of a $T_{i-2}$ to the root of a $T_{i-1}$.




It is easy to see that $|T_i| = F_{i+2} > \phi^i$ where $F_i$ is the $i^{\text{th}}$ *Fibonacci* number and $\phi = (1 + \sqrt{5})/2$.

Our analysis will essentially will show that if a node in a Fibonacci heap has degree $k$, then the tree rooted at that node must include $T_k$. This means that no node can have degree greater than $\log_\phi n$ so $D(n) = O(\log n)$.

Recall that

$$F_k = \begin{cases} 0 & \text{if } k = 0, \\ 1 & \text{if } k = 1, \\ F_{k-1} + F_{k-2} & \text{if } k \geq 2. \end{cases}$$

We will need the following fact

**Lemma:** $\forall k \geq 0, \quad F_{k+2} = 1 + \sum_{i=0}^{k} F_i$.

**Proof:** By induction. Proof is obviously true for $k = 0$. For $k > 0$,

$$\begin{aligned} F_{k+2} &= F_k + F_{k+1} \\ &= F_k + \left(1 + \sum_{i=0}^{k-1} F_i\right) \\ &= 1 + \sum_{i=0}^{k} F_i \end{aligned}$$

**Lemma:**

Let $x$ be any node in a F. heap.

Let $y_1, y_2, \ldots, y_k$ be current children of $x$ in order in which they were linked to $x$.

Then $degree[y_1] \geq 0$ and

$\forall i > 1, \quad degree[y_i] \geq i - 2$.

**Proof:** $degree[y_1] \geq 0$ trivially.

In other cases, when $y_i$ linked to $x$, all of $y_1, y_2, \ldots, y_{i-1}$ were already children so $degree[x] \geq i - 1$.

Node $y_i$ is only linked to $x$ when $degree[y_i] = degree[x]$ so at that time $degree[y_i] \geq i - 1$.

Since then $y_i$ could have lost at most one more child (otherwise it would have been cut and put in root list). So, as long as $y_i$ is linked to $x$, $degree[y_i] \geq i - 2$.

**Lemma:**
Let $x$ be any node in a F. heap and $k = degree[x]$. Then $size(x) =$ number of nodes in tree rooted at $x$ is $\geq F_{k+2} \geq \phi^k$.

**Proof:** Let $s_k$ be the minimum value of $size(x)$ for a node $x$ with degree $k$. It is easy to see that $s_0 = 1$, $s_1 = 2$ and $s_2 = 3$. Also easy to see that $s_i \geq s_{i-1}$.

As before, let $y_1, y_2, \ldots, y_k$ be current children of $x$ in order in which they were linked to $x$

Then, counting $1$ for $x$ and another $1$ for $size(y_1)$,

$$\begin{aligned} size(x) \geq s_k &= 2 + \sum_{i=2}^{k} s_{degree[y_i]} \\ &\geq 2 + \sum_{i=2}^{k} s_{i-2} \end{aligned}$$

We now prove lemma by induction on $s_k \geq F_{k-2}$.

$$\begin{aligned} s_k \geq 2 + \sum_{i=2}^{k} s_{i-2} &\geq 2 + \sum_{i=2}^{k} F_i \\ &= 1 + \sum_{i=0}^{k} F_i = F_{k+2} \end{aligned}$$

We have just proven that if $x$ is any node in a Fibonacci heap and $k = degree[x]$, then $size(x) \geq F_{k+2} \geq \phi^k$.

The intuition behind the proof is that the tree rooted at $x$ must contain the Fibonacci tree $T_i$ defined a few slides back.

An immediate corollary is that

$$size(x) \geq F_{degree(x)+2} \geq \phi^{degree(x)}$$

so no node in an $n$-node Fibonacci heap can have

$$degree \geq \log_\phi n$$

so

$$D(n) = O(\log n).$$