# More on the correspondence between polytopes and LPs.

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Let $$F = \{x \,:\, Ax = b, \quad x \geq 0\}$$

be the feasible region of some LP, $x \in R^n$. Then the corresponding polytope $P$ in $R^{n-m}$ is the solution space to

$$\begin{array}{rcl} b_i - \sum_{j=1}^{n-m} a_{i,j}\, x_j & \geq & 0, \quad i = n-m+1, \ldots, n \\ x_j & \geq & 0, \quad j = 1, \ldots, n-m \end{array}$$

The mapping from $F$ to $P$ is simply

$$\phi((x_1 \ldots, x_n)) = (x_1, \ldots, x_{n-m})$$

Now let $P$ be a polytope in $R^{n-m}$ defined by

$$h_{i,1}\hat{x}_1 + \cdots + h_{i,n-m}\hat{x}_{n-m} + g_i \leq 0, \quad i = 1, \ldots, n.$$

where the first $n-m$ equations are:

$$\hat{x}_i \geq 0, \quad i = 1, \ldots, n-m$$

Then the mapping from $P$ to $F$ is

$$\rho((\hat{x}_1 \ldots, \hat{x}_{n-m})) = (\hat{x}_1 \ldots, \hat{x}_{n-m}, x_{n-m+1}, \ldots, x_n)$$

where

$$x_i = -g_i - \sum_{j=1}^{n-m} h_{i,j}\, x_j, \quad i = n-m+1, \ldots, n.$$

**Lemma:** Using the notation of the previous page.

$$\rho(\phi(F) = F \quad \text{and} \quad \phi(\rho(P)) = P.$$

This just says that $\phi$ and $\rho$ are 1-1 functions.

**Proof:** Next homework.

**Lemma:** Let

$$F = \{x \,:\, Ax = b, \quad x \geq 0\}$$

be the feasible region of a linear program and $P$ the corresponding polytope in $R^{n-m}$.
Now let $c \in R^n$ be a cost vector.

Then there exists a cost vector
$d \in R^{n-m}$ and $K \in R$ such that for every $\hat{x} \in P$

$$K + d'\hat{x} = c'\rho(\hat{x})$$

**Proof:** Next homework.

*Note. This implies that solving the linear program is equivalent to minimizing $d'\hat{x}$ on $P$. This, in turn, is equivalent to sweeping in from infinity the hyperplanes corresponding to $d'\hat{x} = const$ until they hit $P$.*