More on the correspondence between polytopes and LPs.

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Let \( F = \{ x : Ax = b, \ x \geq 0 \} \)

be the feasible region of some LP, \( x \in R^n \). Then the corresponding polytope \( P \) in \( R^{n-m} \) is the solution space to

\[
\begin{align*}
    b_i - \sum_{j=1}^{n-m} a_{i,j} x_j & \geq 0, \quad i = n-m+1, \ldots, n \\
    x_j & \geq 0, \quad j = 1, \ldots, n-m
\end{align*}
\]

The mapping from \( F \) to \( P \) is simply

\[
\phi((x_1, \ldots, x_n)) = (x_1, \ldots, x_{n-m})
\]

Now let \( P \) be a polytope in \( R^{n-m} \) defined by

\[
\begin{align*}
    h_{i,1} \hat{x}_1 + \cdots + h_{i,n-m} \hat{x}_{n-m} + g_i & \leq 0, \quad i = 1, \ldots, n.
\end{align*}
\]

where the first \( n-m \) equations are:

\[
\hat{x}_i \geq 0, \quad i = 1, \ldots, n-m
\]

Then the mapping from \( P \) to \( F \) is

\[
\rho((\hat{x}_1, \ldots, \hat{x}_{n-m})) = (\hat{x}_1, \ldots, \hat{x}_{n-m}, x_{n-m+1}, \ldots, x_n)
\]

where

\[
x_i = -g_i - \sum_{j=1}^{n-m} h_{i,j} x_j, \quad i = n-m+1, \ldots, n.
\]
Lemma: Using the notation of the previous page.

\[ \rho(\phi(F)) = F \quad \text{and} \quad \phi(\rho(P)) = P. \]

This just says that \( \phi \) and \( \rho \) are 1-1 functions.

Proof: Next homework.
**Lemma:** Let

\[ F = \{ x : Ax = b, \quad x \geq 0 \} \]

be the feasible region of a linear program and \( P \) the corresponding polytope in \( \mathbb{R}^{n-m} \).

Now let \( c \in \mathbb{R}^n \) be a cost vector.

Then there exists a cost vector \( d \in \mathbb{R}^{n-m} \) and \( K \in \mathbb{R} \) such that for every \( \hat{x} \in P \)

\[ K + d'\hat{x} = c'\rho(\hat{x}) \]

**Proof:** Next homework.

**Note.** This implies that solving the linear program is equivalent to minimizing \( d'\hat{x} \) on \( P \). This, in turn, is equivalent to sweeping in from infinity the hyperplanes corresponding to \( d'\hat{x} = \text{const} \) until they hit \( P \).