The Primal-Dual Algorithm

P&S Chapter 5

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Simplex solves LP by starting at a Basic Feasible Solution (BFS) and moving from BFS to BFS, always improving the objective function, until no more improvement is possible.

Recall that $x, \pi$ are jointly optimal solutions to primal and dual iff they jointly satisfy the complementary slackness conditions (CSC).

The Primal Dual Algorithm start with a feasible $\pi$ and then iterates the following operations

1. Search for an $x$ that jointly satisfies CSC with $\pi$.

2. If such an $x$ exists, optimality has been achieved. Stop.

3. Otherwise improve $\pi$ and go to 1.
Primal

\[
\begin{align*}
\min z &= c'x \\
Ax &= b \geq 0 \\
x &\geq 0
\end{align*}
\]

Dual

\[
\begin{align*}
\max w &= \pi'b \\
\pi'A &\leq c' \\
\pi' &\geq 0
\end{align*}
\]

We may always assume that \( b \geq 0 \) since, if not, we can multiply appropriate equalities by \(-1\).

We always assume that we know feasible \( \pi \) of dual. If \( c \geq 0 \) we may take \( \pi = 0 \).

If not, we can use a trick due to Beale and introduce
(a) a new variable \( x_{n+1} \) with cost \( c_{n+1} = 0 \).
(b) constraint \( x_1 + x_2 + \cdots + x_n + x_{n+1} = b_{m+1} \)
where \( b_{m+1} \) is large
(c) new dual variable \( \pi_{m+1} \).

New dual is

\[
\begin{align*}
\max w &= \pi'b + \pi_{m+1}b_{m+1} \\
\pi'A_j + \pi_{m+1} &\leq c_j \quad j = 1, \ldots, n \\
\pi_{m+1} &\leq 0
\end{align*}
\]

which has feasible solution

\[
\begin{align*}
\pi_i &= 0 \quad i = 1, \ldots, m \\
\pi_{m+1} &= \min_{1 \leq j \leq n} \{c_j\} < 0
\end{align*}
\]
Assume \( b \geq 0 \) and we know dual feasible \( \pi \).

Recall that \( x, \pi \) are jointly optimal iff they satisfy
\[
\forall i, \pi_i(a_i'x - b_i) = 0 \quad \text{and} \quad \forall j, (c_j - \pi'A_j)x_j = 0.
\]

The Primal-Dual algorithm maintains a feasible \( \pi \).

At each step it solves a Restricted Primal (RP) trying to find a jointly optimal primal solution \( x \). If it doesn’t succeed, that gives the Dual of RP (DRP) enough information to “improve” \( \pi \), while keeping it feasible. This procedure iterates and converges to optimum in a finite number of steps.
Primal \hspace{1cm} Dual
\begin{align*}
\text{min } z &= c'x \\
Ax &= b \geq 0 \\
x &\geq 0
\end{align*}
\begin{align*}
\text{max } w &= \pi'b \\
\pi'A &\leq c' \\
\pi' &\geq 0
\end{align*}

Complementary slackness conditions are
\[ \forall i, \pi_i(a'_i x - b_i) = 0 \quad \text{and} \quad \forall j, (c_j - \pi'A_j)x_j = 0. \]

Define set of admissible columns $J = \{ j : \pi'A_j = c_j \}$.

If $\forall j \notin J$, $x_j = 0$ then $x$ is optimal.

This is equivalent to searching for $x$ that satisfies
\[ \sum_{j \in J} a_{ij}x_j = b_i \quad i = 1, \ldots, m \]
\[ x_j \geq 0 \quad j \in J \]
\[ x_j = 0 \quad j \notin J \]
If we can find $x$ that satisfies equalities below then $x$ is optimal:

$$\sum_{j \in J} a_{ij} x_j = b_i \quad i = 1, \ldots, m$$
$$x_j \geq 0 \quad j \in J$$
$$x_j = 0 \quad j \notin J$$

We therefore introduce $m$ new variables $x_i^a, i = 1, \ldots, m$, and the Restricted Primal (RP)

$$\min \xi = \sum_{i=1}^{m} x_i^a$$
$$\sum_{j \in J} a_{ij} x_j + x_i^a = b_i \quad i = 1, \ldots, m$$
$$x_j \geq 0 \quad j \in J$$
$$(x_j = 0 \quad j \notin J)$$
$$x_i^a \geq 0$$

Solve RP, e.g., using simplex. If optimal solution has $\xi = 0$ then have found optimal $x$ for original problem.

If optimal has $\xi > 0$, consider dual DRP of RP.
RP
\[
\begin{align*}
\min \xi &= \sum_{i=1}^{m} x_i^a \\
\sum_{j \in J} a_{ij} x_j + x^a_i &= b_i \quad i \leq m \\
x_j &\geq 0 \quad j \in J \\
(x_j &= 0 \quad j \notin J) \\
x_i^a &\geq 0 \quad i \leq m 
\end{align*}
\]

Assume that RP has \( \xi > 0 \).
Consider DRP, the dual of RP:

DRP
\[
\begin{align*}
\max w &= \pi^' b \\
\pi^' A_j &\leq 0 \quad j \in J \\
\pi_i &\leq 1 \quad i \leq m \\
\pi_i &\geq 0 \quad i \leq m 
\end{align*}
\]

Let \( \bar{\pi} \) be optimal solution to DRP derived from optimal solution to RP; \( \bar{\pi} = \hat{c}_B B^{-1} \) where \( B \) is basis columns of optimal BFS of RP and \( \hat{c} \) is cost function of RP.
**RP**

\[
\begin{align*}
\text{min } \xi &= \sum_{i=1}^{m} x_i^a \\
\sum_{j \in J} a_{ij} x_j + x_i^a &= b_i \quad i \leq m \\
x_j &\geq 0 \quad j \in J \\
(x_j &= 0 \quad j \notin J) \\
x_i^a &\geq 0
\end{align*}
\]

**DRP**

\[
\begin{align*}
\text{max } w &= \pi'^{b} \\
\pi' A_j &\leq 0 \quad j \in J \\
\pi_i &\leq 1 \quad i \leq m \\
\pi_i &\geq 0
\end{align*}
\]

\[J = \{j : \pi' A_j = c_j\}\]

In original Dual

Started with feasible \(\pi\). Using \(\text{RP}\), tried to find \(x\) that jointly satisfied CSC with \(\pi\).

Optimum \(\xi_0 > 0\), so this didn’t exist, but we can find \(\bar{\pi}\), optimum of \(\text{DRP}\). Idea is to try and improve \(\pi\) to \(\pi^*\) by finding “good” \(\theta\) to set

\[\pi^* = \pi + \theta \bar{\pi}.\]

Cost of \(\pi^*\) is

\[\pi'^* b = \pi'^b + \theta \bar{\pi}' b.\]

Since RP and DRP are a primal-dual pair we have \(\bar{\pi}' b = \xi_{opt} > 0\). Therefore, to improve \(\pi\) to \(\pi^*\), we must have \(\theta > 0\).
Dual
\[\begin{align*}
\max w &= \pi'^b \\
\pi'A &\leq c' \\
\pi' &\geq 0
\end{align*}\]
\[J = \{j : \pi' A_j = c_j\}\]
\(\pi\) is feasible

DRP
\[\begin{align*}
\max w &= \pi'^b \\
\pi' A_j &\leq 0 \quad j \in J \\
\pi_i &\leq 1 \quad i \leq m \\
\pi_i &\geq 0
\end{align*}\]
\(\bar{\pi}\) is optimal

We “improve” cost of \(\pi\) by setting
\[\pi'^* = \pi + \theta \bar{\pi}, \quad \theta > 0.\]

In order to maintain feasibility of \(\pi\) we need
\[\forall j, \quad \pi'^* A_j = \pi' A_j + \theta \bar{\pi}' A_j \leq c_j.\]

If \(\bar{\pi}' A_j \leq 0\) this is not a problem.

In particular, if \(\bar{\pi}' A_j \leq 0\) for all \(j\), then \(\theta\) can be made arbitrarily large so original dual is unbounded and original primal is infeasible.
Dual

\[
\begin{align*}
\text{max } w &= \mathbf{\pi'}b \\
\mathbf{\pi'}A &\leq c' \\
\mathbf{\pi'} &\geq 0 \\
J &= \{j : \mathbf{\pi'}A_j = c_j\}
\end{align*}
\]

DRP

\[
\begin{align*}
\text{max } w &= \mathbf{\pi'}b \\
\mathbf{\pi'}A_j &\leq 0 \quad j \in J \\
\mathbf{\pi}_i &\leq 1 \quad i \leq m \\
\mathbf{\pi}_i &\geq 0 \\
\bar{\mathbf{\pi}} \text{ is optimal}
\end{align*}
\]

We just saw that if, \(\forall j, \bar{\mathbf{\pi}}'A_j \leq 0\), then original primal is infeasible.

We know that \(\forall j \in J, \bar{\mathbf{\pi}}'A_j \leq 0\) since \(\bar{\mathbf{\pi}}\) is optimal and thus feasible. Then

**Theorem** If \(\xi_{opt} > 0\) in RP and the optimal dual (in DRP) satisfies

\[
\bar{\mathbf{\pi}}'A_j \leq 0 \quad \text{for } j \notin J
\]

then \(P\) is infeasible.
In order to maintain \textit{feasibility} of $\pi$ we need
\[ \forall j, \quad \pi^* A_j = \pi' A_j + \theta \bar{\pi}' A_j \leq c_j. \]

From previous slide, when maintaining feasibility, we only worry about $\bar{\pi}' A_j > 0$ for some $j \notin J$. i.e.,
\[ \pi^* A_j = \pi' A_j + \theta \bar{\pi}' A_j \leq c_j \quad j \notin J \text{ and } \bar{\pi}' A_j > 0 \]

\textbf{Theorem:} When $\xi_{opt} > 0$ in RP and there is a $j \notin J$ with $\bar{\pi}' A_j > 0$, the largest $\theta$ that maintains the feasibility of $\pi^* = \pi + \theta \bar{\pi}$ is
\[ \theta_1 = \min_{\substack{j \notin J \ s.t. \ \bar{\pi}' A_j > 0}} \left[ \frac{c_j - \pi' A_j}{\bar{\pi}' A_j} \right] \]

The new cost is
\[ w^* = \pi' b + \theta_1 \bar{\pi}' b = w + \theta_1 \bar{\pi}' b > w. \]
**procedure** primal-dual

begin

infeasible := ‘no’, opt := ‘no’;
let \( \pi \) be feasible in \( D \)

while infeasible = ‘no’ and opt = ‘no’ do

begin set \( J = \{ j : \pi'A_j = c_j \} \);

solve RP by the simplex algorithm;

if \( \xi_{opt} = 0 \) then opt := ‘yes’

else if \( \bar{\pi}'A_j \leq 0 \) for all \( j \notin J \)

then infeasible := ‘yes’

else \( \pi := \pi + \theta_1 \bar{\pi} \)

(comment: \( \theta_1 \) from last slide)

end

end

end

Note: recall that \( \bar{\pi} = \hat{c}_B B^{-1} \) is optimal solution to DRP derived from optimal solution to RP, where \( B \) is composed of basis columns of optimal BFS of RP and \( \hat{c} \) is cost function of RP.
Quick Review of Relative Cost

Recall that given an LP in standard form and its dual D then

1. Let $B$ be a BFS of the LP $Ax = b$, $x \geq 0$ and $c_B'$ the associated cost vector. For all $j$, the relative cost of $x_j$ is

   $$\bar{c}_j = c_j - z_j = c_j - c_B'B^{-1}A_j$$

2. If $B$ is an optimal BFS then we can choose $\pi' = c_B'B^{-1}$ as an optimal solution to Dual.

3. If $x$ is an optimal BFS and $\pi' = c_B'B^{-1}$ is the associated optimal dual solution, then

   $$\bar{c}_j = c_j - z_j = c_j - \pi'A_j.$$
Recall that \( J = \{ j : \pi' A_j = c_j \} \). We now claim:

**Theorem:** Every admissible column in the optimal basis of RP remains admissible at the start of the next iteration.

**Proof:** Suppose column \( j \) is in the optimal basis of RP at the end of an iteration. Then its relative cost (in RP) is

\[
0 = \overline{c}_j = -\overline{\pi}' A_j.
\]

This means that

\[
\pi^* A_j = \pi' A_j + \theta_1 \overline{\pi}' A_j = \pi' A_j = c_j
\]

so \( j \) remains in \( J \).

One consequence is that if at some iteration RP has an optimal BFS \( \hat{x}_0 \) then, at the start of the next iteration, \( \hat{x}_0 \) remains a BFS (although probably no longer optimal) *in the new RP*. We may therefore start simplex in the new RP at the old optimal solution.
Recall the definition of 

$$\theta_1 = \min_{j \notin J} \left[ \frac{c_j - \pi'^{\prime}A_j}{\pi'^{\prime}A_j} \right]$$

Let $j = j_0$ be value at which minimum occurs. Then 

$$\pi'^{\prime}A_{j_0} = \pi'^{\prime}A_{j_0} + \theta_1 \pi'^{\prime}A_{j_0} = c_{j_0}$$

so $j_0$ enters $J$.

At the end of the previous iteration it is possible that $j_0$ could not enter the BFS because it was not in $J$ and was therefore not considered.

Now that $j_0 \in J$, it might be able to enter a BFS. Let $\hat{x}_0$ be current optimal BFS of RP at end of last iteration. It remains a BFS in the new RP.

Since $\bar{\pi}'A_j > 0$ we have (from page 13 (1,2)) and fact that $\tilde{c}_j = 0$) that, for BFS $\hat{x}_0$, the relative cost of $x_{j_0}$ in the new RP is $-\pi'^{\prime}A_{j_0} < 0$.

We can therefore pivot on $x_{j_0}$ and (if the BFS is not degenerate) we will improve the cost.
Consider a primal $P'$ where the words (i) $j \in J$ and (ii) $x_j = 0$, $j \notin J$ were deleted. Any BFS of RP would be a BFS of $P'$, so there are only a finite number of BFSs shared among all of the RPs.
We can consider an iteration of the primal-dual algorithm as starting from some BFS of $P'$ which is a BFS of our current RP and finding a pivot $j_0$ to start moving to another BFS of $P'$ which is also BFS of our current RP. If the pivot is not degenerate then our cost will decrease. (If the pivot is degenerate we use an anti-cycling rule to guarantee that we will not cycle). Then, since our algorithm moves from BFS to BFS without ever repeating a BFS, it must terminate. When it terminates, it either shows infeasibility of original problem $P$ or reaches optimality of $P$.

**Next:** The reason that this is an interesting technique is that we will soon see that solving RP and/or DRP can often be done using other combinatorial algorithms.