MAX-SAT: Best of Two

We have so far seen two different approaches to approximating MAX-SAT:

• **Random MAX-SAT.**
  \[ E(W) \geq \frac{OPT}{2}. \]
  This chose a random truth assignment using a fair coin. For clause \( C_j \) with length \( l_j \)
  \[ \Pr(\text{\(C_j\) is satisfied}) = 1 - 2^{-l_j}. \]

• **Randomized Rounding.**
  \[ E(W) \geq (1 - \frac{1}{e}) \cdot OPT \approx 0.632 \cdot OPT. \]
  Finds solution \((y^*, x^*)\) to relaxed linear program. For clause \( C_j \) with length \( l_j \)
  \[ \Pr(\text{\(C_j\) is satisfied}) \geq \left( 1 - \left( 1 - \frac{1}{l_j} \right)^{l_j} \right) z_j^*. \]

Notice that **Random MAX-SAT** is “good” for long clauses while **Randomized Rounding** is “good” for short clauses. We will now see how to combine the two to get an even better approximation.
The *Best of Two* algorithm is to run *Random MAX-SAT* to get assignment $x^1$ with weight $W_1$ and to also run *Randomized Rounding* to get assignment $x^2$ with weight $W_2$. Then compare $W_1$ and $W_2$. If $W_1 > W_2$ return $x^1$, else return $x^2$. Let $W$ be the weight of the returned assignment.

**Lemma:** $E(W) \geq \frac{3}{4} OPT$.

**Proof:** We use the fact that

$$W = \max(W_1, W_2) \geq \frac{1}{2}W_1 + \frac{1}{2}W_2.$$  

Therefore

$$E(W) \geq E\left(\frac{1}{2}W_1 + \frac{1}{2}W_2\right)$$

$$= \frac{1}{2}E(W_1) + \frac{1}{2}E(W_2)$$

$$= \frac{1}{2} \sum_j w_j Pr(C_j \text{ is satisfied by } x^1)$$

$$+ \frac{1}{2} \sum_j w_j Pr(C_j \text{ is satisfied by } x^2)$$

$$\geq \frac{1}{2} \sum_j w_j \left(1 - 2^{-l_j}\right)$$

$$+ \frac{1}{2} \sum_j w_j \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_j^*$$

$$= \sum_j w_j \left(\frac{1}{2} \left(1 - 2^{-l_j}\right) + \frac{1}{2} \left(1 - \left(1 - \frac{1}{l_j}\right)^{l_j}\right) z_j^*\right)$$

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So far we have seen that

\[ E(W) \geq \sum_j w_j \left( \frac{1}{2} (1 - 2^{-l_j}) + \frac{1}{2} \left( 1 - \left( 1 - \frac{1}{l_j} \right)^{l_j} \right) \right) z_j^* \].

We will now show that, for all \( j \),

\[ \frac{1}{2} (1 - 2^{-l_j}) + \frac{1}{2} \left( 1 - \left( 1 - \frac{1}{l_j} \right)^{l_j} \right) z_j^* \geq \frac{3}{4} z_j^* \].

This will imply that

\[ E(W) \geq \sum_j \frac{3}{4} w_j z_j^* \geq \frac{3}{4} \text{OPT} \]

and we will be done.

We prove this case by case.
If \( l_j = 1 \) then \( \frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} z_j^* \geq \frac{3}{4} z_j^* \).

If \( l_j = 2 \) then \( \frac{1}{2} \cdot \frac{3}{4} + \frac{1}{2} \cdot \frac{3}{4} z_j^* \geq \frac{3}{4} z_j^* \).

If \( l_j \geq 3 \) then

\[ \frac{1}{2} (1 - 2^{-l_j}) + \frac{1}{2} \left( 1 - \left( 1 - \frac{1}{l_j} \right)^{l_j} \right) z_j^* \geq \frac{1}{2} \cdot \frac{7}{8} + \frac{1}{2} \left( 1 - \frac{1}{e} \right) z_j^* \geq \frac{3}{4} z_j^* \].

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