Dynamic Programming: The Rod Cutting Problem

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Dynamic Programming (DP) bears similarities to Divide and Conquer (D&C)

- Both partition a problem into smaller subproblems and build solution of larger problems from solutions of smaller problems.
- In D&C, work top-down. Know exact smaller problems that need to be solved to solve larger problem.
- In D, (usually) work bottom-up. Solve all smaller size problems and build larger problem solutions from them.
  - In DP, many large subproblems reuse solution to same smaller problem.

- DP often used for optimization problems
- Problems have many solutions; we want the best one

Main idea of DP

1. Analyze the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution (usually bottom-up)
Rod Cutting

- **Input:** We are given a rod of length \( n \) and a table of prices \( p_i \) for \( i = 1, \ldots, n; \) \( p_i \) is the price of a rod of length \( i \).

- **Goal:** to determine the maximum revenue \( r_n \), obtainable by cutting up the rod and selling the pieces.

- **Example:** \( n = 4 \) and \( p_1 = 1, p_2 = 5, p_3 = 8, p_4 = 9 \)
  - If we do not cut the rod, we can earn \( p_4 = 9 \)
  - If we cut it into 4 pieces of length 1, we earn \( 4 \cdot p_1 = 4 \)
  - If we cut it into 2 pieces of length 1 & a piece of length 2, we earn \( 2 \cdot p_1 + p_2 = 9 \)
  - If we cut it into 2 pieces of length 2, we can earn \( 2 \cdot p_2 = 10 \)
  - There are more options, but the maximum revenue is 10

- In general, rod of length \( n \) can be cut in \( 2^{n-1} \) different ways, since we can choose cutting, or not cutting, at all distances \( i \) \( (1 \leq i \leq n - 1) \) from the left end.
Optimal Solution

- We can calculate the maximum revenue \( r_n \) in terms of optimal revenues for shorter rods

\[
r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1)
\]

- \( p_n \) if we do not cut at all
- \( r_1 + r_{n-1} \) if we take the sum of optimal revenues for 1 and \( n-1 \)
- \( r_2 + r_{n-2} \) if we take the sum of optimal revenues for 2 and \( n-2 \)
- \( \ldots \)

- Another approach. Set \( r_0 = 0 \) and

\[
r_n = \max_{1 \leq i \leq n} (p_i + r_{n-i})
\]

- Cut a piece of length \( i \), with remainder of length \( n - i \)
- Only the remainder, and not the first piece, may be further divided
Recursive Top-down Implementation

Cut-Rod\((p, n)\)

\[
\text{if } n = 0 \text{ then}
\]
\[
\quad \text{return } 0;
\]
\[
\text{end}
\]
\[
q = -\infty;
\]
\[
\text{for } i = 1 \text{ to } n \text{ do}
\]
\[
\quad q = \max(q, p[i] + \text{Cut-Rod}(p, n - i));
\]
\[
\text{end}
\]
\[
\text{return } q;
\]

Algorithm Time

- \(T(n)\): the total number of calls made to Cut-Rod when called with rod length \(n\)

\[
T(n) = \begin{cases} 
1 + \sum_{0 \leq j \leq n-1} T(j), & \text{if } n > 0, \\
1, & \text{if } n = 0.
\end{cases}
\]

- Induction \(\Rightarrow T(n) = 2^n\)
Algorithm calls same subproblem many times
After solving a subproblem, store the solution
- Next time you encounter same subproblem, lookup the solution, instead of solving it again
- Uses space to save time

Two main methodologies: top-down and bottom-up
- Corresponding algorithms have the same asymptotic cost, but bottom-up is usually faster in practice

Main idea of bottom-up DP
- Don’t wait until until subproblem is encountered.
- Sort the subproblems by size; solve smallest subproblems first
- Combine solutions of small subproblems to solve larger ones
- $p_i$ are the problem inputs.
- $r_i$ is max profit from cutting rod of length $i$.
- Goal is to calculate $r_n$
- $r_i$ defined by
  - $r_1 = 1$ and $r_n = \max_{1 \leq i \leq n}(p_i + r_{n-i})$
- Iteratively fill in $r_i$ table by calculating $r_1, r_2, r_3, \ldots$
- $r_n$ is final solution

<table>
<thead>
<tr>
<th>$i$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>...</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_i$</td>
<td>$p_1$</td>
<td></td>
<td></td>
<td></td>
<td>...</td>
<td></td>
</tr>
</tbody>
</table>
Bottom-Up-Cut-Rod($p, n$)

```python
r[0] = 0; // Array $r[0...n]$ stores the computed optimal values
for j = 1 to n do
    // Consider problems in increasing order of size
    q = -\infty;
    for i = 1 to j do
        // To solve a problem of size $j$, we need to consider all
decompositions into $i$ and $j - i$
        q = max(q, $p[i] + r[j - i]$);
    end
    r[j] = q;
end
return $r[n]$;
```

- **Cost:** $O(n^2)$
  - The outer loop computes $r[1], r[2], \ldots, r[n]$ in this order
  - To compute $r[j]$, the inner loop uses all values $r[0], r[1], \ldots, r[j - 1]$ (i.e., $r[j - i]$ for $1 \leq i \leq j$)
Algorithm only computes \( r_i \). It does not output the cutting.

Easy fix

- When calculating \( r_j = \max_{1 \leq i \leq j}(p_i + r_{j-i}) \)
  - store value of \( i \) that achieved this max in new array \( s[j] \).
  - This \( j \) is the size of last piece in the optimal cutting.

After algorithm is finished, can reconstruct optimal cutting by unrolling the \( s_j \).
Extended-Bottom-Up-Cut-Rod\((p,n)\)

\[
\begin{align*}
\text{// Array } s[0\ldots n] & \text{ stores the optimal size of the first piece to cut off} \\
r[0] & = 0; \quad \text{// Array } r[0\ldots n] \text{ stores the computed optimal values} \\
\text{for } j = 1 \text{ to } n \text{ do} \\
& \quad q = -\infty; \\
& \quad \text{for } i = 1 \text{ to } j \text{ do} \\
& \quad \quad \text{// Solve problem of size } j \\
& \quad \quad \text{if } q < p[i] + r[j - i] \text{ then} \\
& \quad \quad \quad q = p[i] + r[j - i]; \\
& \quad \quad \quad s[j] = i; \quad \text{// Store the size of the first piece} \\
& \quad \quad \text{end} \\
& \quad r[j] = q; \\
\text{end} \\
\text{while } n > 0 \text{ do} \\
& \quad \text{// Print sizes of pieces} \\
& \quad \text{Print } s[n]; \\
& \quad n = n - s[n]; \\
\text{end}
\end{align*}
\]