An Introduction to Hashing

(Following CLRS)
Outline

- Introduction
- Hashing with Chaining
- Open Addressing
- Hash Functions & Universal Hashing
- Odds & Ends
Known: A set \( U = \{s, 1, 2, \ldots, u - 1\} \) of the universe of possible keys that could exist.

Goal: To maintain a dictionary that permits the following operation on keys

- **Search\( (x) \)**: Find the record with key \( x \) or report that it does not exist
- **Insert\( (x) \)**: Insert a new record with key \( x \)
- **Delete\( (x) \)**: Delete the record with key \( x \)
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Would like $O(1)$ (average) time per operation.
Introduction (ii)

Universe Size: $U$
Number of actual keys: $n$ ($n << U$)
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For now, assume \textit{uniform hashing}, that, every key is equally likely to hash to any of the \( m \) slots,

\[ \forall x, i, \quad \Pr (h(x) = i) = \frac{1}{m}. \]
Introduction (iii)

$h : U \rightarrow \{0, 1 \ldots, m - 1\}$

$h$ maps the set of keys into a “small” table. Key $k$ is stored in table slot $h(k)$.

Finding key $k$ is then just a matter of going to table location $h(k)$.

Problem is that, since $m$ is small, many keys might be mapped to same slot, creating collision.
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Problem is that, since \( m \) is small, many keys might be mapped to same slot, creating collision.

Two major approaches to addressing collisions:
(1) Chaining
(2) Open Addressing
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Chaining

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

All elements that hash to the same slot are put into the same linked list.
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Diagram:

- \( k_4 \rightarrow k_1 \)
- \( k_6 \rightarrow k_3 \)
- \( k_2 \)
- \( k_5 \)
Chaining

\[ h : U \rightarrow \{0, 1\ldots, m - 1\} \]

All elements that hash to the same slot are put into the same linked list.

```
    k_7 -> k_4 -> k_1
    
    k_6 -> k_3
    
    k_2
    
    k_5
```
Chaining

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

All elements that hash to the same slot are put into the same linked list

**Insert(x):** Insert \( x \) into front of list for slot \( h(x) \)

**Delete(x):** Delete \( x \) from list for slot \( h(x) \), if it’s there.

*Use doubly linked lists*

**Search(x):** Search for \( x \) in list for \( h(x) \)
Chaining

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

All elements that hash to the same slot are put into the same linked list.

- **Insert(x):** Insert \( x \) into front of list for slot \( h(x) \)  \( O(1) \)
- **Delete(x):** Delete \( x \) from list for slot \( h(x) \), if it’s there.  
  *Use doubly linked lists  \( O(1) \)*
- **Search(x):** Search for \( x \) in list for \( h(x) \)  \( O(\text{length of list}) \)
Chaining: Unsuccessful Search

\( h : U \rightarrow \{0, 1 \ldots, m - 1\} \)

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Search(x): Search for \( x \) in list for \( h(x) \) \( O(\text{length of list}) \)

Recall load factor \( \alpha = \frac{n}{m} \).
This is average \# items per list.

Unsuccessful search for \( x \) not in table will require searching entire list for \( h(x) \).

Worst case length is \( O(1) \).
Average case length is \( O(\alpha) \).
Chaining: Unsuccessful Search

$h : U \rightarrow \{0, 1 \ldots, m - 1\}$

Search($x$): Search for $x$ in list for $h(x)$ $O(\text{length of list})$

Recall load factor $\alpha = \frac{n}{m}$. This is average $\#$ items per list.

Unsuccessful search for $x$ not in table will require searching entire list for $h(x)$.

Worst case length is $O(1)$.

Average case length is $O(\alpha)$.

Average Unsuccessful Search time is $O(1 + \alpha)$ where 1 is amount of time to calculate $h(x)$. 
Chaining: Successful Search

For Successful Search for $x$:
Assume $x$ equally likely to be any item in table
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Search cost is $\#$ items ahead of $x$ in list $h(x)$

$$= \# \text{ of items inserted into } h(x) \text{ after } x$$
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If $x$ is $i$’th item inserted
$\Rightarrow$ $n - i$ items inserted after $x$

$\Rightarrow \alpha - \frac{i}{m} = \frac{n-i}{m}$ items inserted on average into $h(x)$ after $x$

$x$ is equally likely (with prob $1/n$) to be $i$’th inserted item.

Average $\#$ of items ahead of $x$ in list $h(x)$ is

$$\frac{1}{n} \sum_{i=1}^{n} \left( \alpha - \frac{i}{m} \right) = \alpha - \frac{n(n+1)}{2nm} = \alpha - \frac{\alpha}{2} + \frac{\alpha}{2n}$$
Chaining: Successful Search

For Successful Search for \( x \):
Assume \( x \) equally likely to be any item in table
Search cost is \( \# \) items ahead of \( x \) in list \( h(x) \)
\[
\begin{align*}
= \# \text{ of items inserted into } h(x) \text{ after } x
\end{align*}
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If \( x \) is \( i' \)th item inserted

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\( x \) is equally likely (with prob \( 1/n \)) to be \( i' \)th inserted item.

Average \( \# \) of items ahead of \( x \) in list \( h(x) \) is

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\frac{1}{n} \sum_{i=1}^{n} \left( \alpha - \frac{i}{m} \right) = \alpha - \frac{n(n+1)}{2nm} = \alpha - \frac{\alpha}{2} + \frac{\alpha}{2n}
\]

Adding 1 unit of time to calculate \( h(x) \)

Average cost of successful search is \( \Theta(1 + \alpha) \).
Chaining

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

Search(\(x\)): Search for \(x\) in list
for \(h(x)\) \(O(\text{length of list})\)

Both Successful and Unsuccessful Search require \(O(1 + \alpha)\) time on average

where \(\alpha = \frac{n}{m}\) is the load factor
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Open Addressing

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

- No lists. All keys stored in hash table itself.
- For insertion, *probe* hash table until empty slot for insertion is found.
- *Probe Sequence* is part of hash function.
- Hash function is now
  \[ h : U \times \{0, 1, \ldots, m - 1\} \rightarrow \{0, 1, \ldots, m - 1\} \]
- Probe sequence for \( x \) is,
  \[ h(x, 0), h(x, 1), \ldots, h(x, m - 1) \]
  which is a permutation of \( \{0, 1, \ldots, m\} \)
- For search(x), *probe* hash table using probe sequence for \( h(x) \) until either \( x \) or empty slot for insertion is found.
Open Addressing: Linear Probing

$h' : U \rightarrow \{0, 1 \ldots, m - 1\}$

- **Hash Function** is $h(x, i) = (h'(x) + i) \mod m$
  where $g'(x)$ is original hash function.

- **Insert**: Attempts insertion at $h'(x)$, then $h'(x) + 1$, $h'(x) + 2$, etc., (wrapping around to 0 after reaching end of table) until empty slot is found and $x$ inserted there.

- **Search(x)**: Examines probe sequence until it finds $x$ or an empty slot.
  If empty slot is found then $x$ wasn’t previously inserted and search unsuccessful

- **Deletion**: More complicated.
Open Addressing: Linear Probing

\( h' : U \rightarrow \{0, 1 \ldots, m-1\} \)

- Hash Function is \( h(x, i) = (h'(x) + i) \mod m \) where \( g'(x) \) is original hash function.
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- **Search\( (x) \)**: Examines probe sequence until it finds \( x \) or an empty slot.
  If empty slot is found then \( x \) wasn’t previously inserted and search unsuccessful.
- **Deletion**: More complicated.
  
  *Can’t actually delete item and reset slot as ‘empty’*
  *That would mess up Search\( (x) \).*
  *Can mark slot as (used but) deleted.*
  *Deletion in open addressing does cause difficulties.*
  *Better to use chaining.*
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).
Open Addressing: Linear Probing

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As example, let \( h(x) = x \mod m \) with \( m = 12 \).

*Only for illustration. This is a BAD hash function.*
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Insert(15)
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Open Addressing: Linear Probing

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\[
\begin{array}{c}
0 \\
1 \\
15 \\
3 \\
\vdots \\
35 \\
\end{array}
\]

- Insert(15)
- Insert(0)
- Insert(35)
- Insert(3)
- Insert(11)
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Insert(35)
Insert(3)
Insert(11)

\begin{array}{c|c}
0 & 0 \\
1 & \\
15 & 15 \\
3 & 3 \\
35 & 35 \\
\end{array}
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

\begin{verbatim}
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\end{verbatim}

Only for illustration. This is a BAD hash function.

Insert(15)
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Insert(3)
Insert(11)
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \ mod \ m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad \quad \quad h(x) = (h'(x) + 1) \mod m \]

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As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Open Addressing: Linear Probing

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| \text{Search(11)} |  \\
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As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Open Addressing: Linear Probing

\[ h' : U \to \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

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Search(11)
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\[ \text{Search}(11) \]
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Search(11) Exists
**Open Addressing: Linear Probing**

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- Search(11)  Exists
- Search(3)    

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Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

*Only for illustration. This is a BAD hash function*

Search(11) \hspace{1cm} Exists

Search(3)
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Open Addressing: Linear Probing

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Search(11) \hspace{1cm} Exists

Search(3) \hspace{1cm} Exists

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Open Addressing: Linear Probing

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Search(3) \hspace{1cm} Exists
Search(9) \hspace{1cm} Does not exist
Search(24)
Open Addressing: Linear Probing

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Easy to code but suffers from primary clustering.

Long runs build up, increasing average search time
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Easy to code but suffers from primary clustering.
Long runs build up, increasing average search time

One fix is to change probe sequence to no longer be linear.
Open Addressing: Quadratic Probing

$h' : U \rightarrow \{0, 1 \ldots, m - 1\}$

- Hash Function is $h(x, i) = (h'(x) + c_1 + c_2i^2) \mod m$
  where $h'(x)$ is original hash function and $c_1, c_2$ fixed constants.
- As example we will set $h'(x) = x \mod 12$, $c_1 = 0$ and $c_2 = 1$ so
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Open Addressing: Double Hashing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \]

- Hash Function is \( h(x, i) = (h_1(x) + ih_2(x)) \mod m \)
- \( h_1(x) \) and \( h_2(x) \) are auxiliary hash functions
- Note that (unlike before) probe sequence depends upon \( x \)
- In order for probe sequence to check entire table, must have \( h_2(x) \) be relatively prime to \( m \), e.g.,
  - \( m \) a power of 2; \( h_2(x) \) always odd
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Example: \( m = 13 \)
\[
\begin{align*}
  h_1(x) &= k \mod m \\
  h_2(x) &= 1 + (k \mod 11)
\end{align*}
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14 would have probe sequence 1, 5, 9, \ldots  Since first 2 locations full, it will be inserted into 9.
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\]

14 would have probe sequence 1, 5, 9, \ldots
Since first 2 locations full, it will be inserted into 9.
Open Addressing: Analysis Results

We have seen 3 different open addressing collision resolution methods:

- **Linear Probing:**  \( h(x, i) = (h'(x) + 1) \mod m \)
- **Quadratic Probing**  \( h(x, i) = (h'(x) + c_1 + c_2x^2) \mod m \)
- **Double Hashing**  \( h(x, i) = (h_1(x) + ih_2(x)) \mod m \)
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For analysis, we often assume **uniform hashing**. This states that the probe sequence

\[
\begin{align*}
h(x, 0), h(x, 1), h(x, 2), \ldots, h(x, m)
\end{align*}
\]

is equally likely to be any of the \( m! \) permutations of \( 1, 2, \ldots, m \).
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\[ h(x, 0), h(x, 1), h(x, 2), \ldots, h(x, m) \]

is equally likely to be any of the \( m! \) permutations of \( 1, 2, \ldots, m \).

Uniform Hashing is not actually realizable. The more random our probe sequence, though, the closer actual behavior is to theory.
Open Addressing: Analysis Results

Recall $\alpha = \frac{n}{m}$ is the load factor.
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Lemma: Given an open-address hash table with load factor $\alpha = n/m < 1$, the average number of probes in an unsuccessful search is at most

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Open Addressing: Analysis Results

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**Lemma:** Given an open-address hash table with load factor $\alpha = \frac{n}{m} < 1$, the average number of probes in an unsuccessful search is at most $\frac{1}{1 - \alpha}$

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$$\frac{1}{\alpha \ln \frac{1}{1 - \alpha}}$$
Outline

• Introduction
• Hashing with Chaining
• Open Addressing
• Hash Functions & Universal Hashing
• Odds & Ends
Hash Functions & Universal Hashing

Returning to chained hashing, notice that our analysis assumed that the hashed keys were equally distributed among the slots.

- If all keys hashed to same slot, performance would be very bad.
- If the hash function \( h(x) \) is given in advance and \( n << U \), very easy to construct bad case in which all keys map to the same slot.

- We sidestep this issue by choosing a random hash function.
- More specifically, we will have a collection of hash functions \( \mathcal{H} \).
- Given any set of keys, we will choose a random hash function \( h \in \mathcal{H} \) and then hash using \( h(x) \).

- On average, the \( n \) set of keys will be hashed so that each slot will get \( O(n/m) = O(\alpha) \) keys.

- Our \( O(1 + \alpha) \) successful/unsuccessful search times for chained hashing will then hold on average.

- One class \( \mathcal{H} \) of hash functions having this property are the Universal ones; they permit Universal Hashing.
Universal Hashing

• Let $\mathcal{H}$ be a set of hash functions, such that each $h \in \mathcal{H}$ maps $h : U \rightarrow \{0, 1, \ldots, m - 1\}$

• $\mathcal{H}$ is Universal if, for every two different keys $k, \ell$, the number of hash functions in $\mathcal{H}$ that map $k, \ell$ to the same slot is at most $|\mathcal{H}|/m$, i.e.,

$$\forall k \neq \ell \in U, \quad |\{h \in \mathcal{H} : h(k) = h(\ell)\}| \leq \frac{|\mathcal{H}|}{m}.$$
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Let \( k_1, k_2, \ldots, k_n \) be the \( n \) keys. Let \( i \) be any fixed index. Then, for \( j \neq i \), if \( h \in \mathcal{H} \) is chosen uniformly at random,

\[
\Pr(h(k_i) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \cdot \frac{|\mathcal{H}|}{m} = \frac{1}{m}.
\]

From linearity of expectation, if \( h \in \mathcal{H} \) is chosen uniformly at random, average \# of other keys mapping to the same slot as \( k_i \) is then

\[
\sum_{j \neq i; 1 \leq j \leq n} \Pr(h(k_i) = h(k_j)) \leq \frac{n - 1}{m} < \alpha
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Let $k_1, k_2, \ldots, k_n$ be the $n$ keys. Let $i$ be any fixed index. Then, for $j \neq i$, if $h \in \mathcal{H}$ is chosen uniformly at random,

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Similarly, if $k$ is not one of the $n$ keys then, for all $j$,

\[
\Pr(h(k) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{m} = \frac{1}{m} \quad \text{and average $\#$ of keys mapping to same slot as $k$ is} \quad \sum_{j=1}^{n} \Pr(h(k) = h(k_j)) \leq n/m = \alpha
\]
Construction of Universal Hash Functions

- Choose prime $p > U$
- Set $Z^*_p = \{1, 2, 3, \ldots, p-1\}$ and $Z_p = \{0, 1, 2, 3, \ldots, p-1\}$
- Define

$$\forall a \in Z^*_p, b \in Z_p, \quad h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m$$
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Example: Set \( p = 17, \ m = 6 \). Then

\[
h_{3,4}(8) = ((3 \cdot 8 + 4) \mod 17) \mod 6 = 5
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Lemma: The Class $\mathcal{H} = \{ h_{a,b} : a \in Z_p^*, b \in Z_p \}$ is Universal.

Proof: Need to show that for all $k \neq \ell$, number of pairs $(a, b)$ with $h_{a,b}(k) = h_{a,b}(\ell)$ is $\leq p(p - 1)/m$
Construction of Universal Hash Functions (ii)

∀a ∈ Z∗ p, b ∈ Zp,  \[ h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m \]

\( p \) prime, \( Z^*_p = \{1, 2, 3, \ldots, p-1\} \), \( Z_p = \{0, 1, 2, 3, \ldots, p-1\} \)
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(1) Let \( k \neq \ell \in U \). For given \((a, b) \in \mathbb{Z}_p^* \times \mathbb{Z}_p\) set

\[ r = (ak + b) \mod p, \quad s = (a\ell + b) \mod p \]
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\[
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\]

(2) Every different \( (a, b) \) pair generates a unique \( (r, s) \) pair

For a given \( (r, s) \) pair we can solve
\[
    a = (r - s)(k - \ell)^{-1} \mod p, \quad b = (r - ak) \mod p.
\]

where \( (k - \ell)^{-1} \) is the multiplicative inverse base \( p \). Since, for fixed \( p, k, \ell \), we must have \( r \neq s \), there are are \( p(p - 1) \) \( (r, s) \) pairs. Since there are also \( p(p - 1) \) \( (a, b) \) pairs, there is a one-one correspondence between them, with every \( (a, b) \) pair generating a different \( (r, s) \).
Construction of Universal Hash Functions (iii)

\[ \forall a \in \mathbb{Z}_p^{*}, b \in \mathbb{Z}_p, \quad h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m \]

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Summing over all p possible values of r gives \leq \frac{p(p - 1)}{m}

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pairs \((r, s)\) with \(s \neq r\) and \(r \equiv s \mod m\),
i.e., \(\leq p(p - 1)/m = |\mathcal{H}|/m\) pairs \((a, b)\) with \(h_{a,b}(k) = h_{a,b}(\ell)\)
\(\Rightarrow \mathcal{H}\) is Universal
Universal Hashing: Wrap Up

- Just saw that the set of Hash functions

\[ \mathcal{H} = \{ h_{a,b} : a \in \mathbb{Z}^*_p, b \in \mathbb{Z}_p \} \]

is \textit{Universal}

- This implies that for any set of \( n \) keys \( K = \{ k_1, k_2, \ldots, k_n \} \), an effective way of storing the keys is to
  - Choose a random pair \((a, b)\) uniformly at random from the \( p(p-1) \) pairs in \( \mathbb{Z}_p^* \times \mathbb{Z}_p \)
  - Hash the items in \( K \) using hash function \( h_{a,b} \)

- Because \( \mathcal{H} \) is Universal, average time for storing the data will be \( O(n\alpha) \) where \( \alpha = n/m \) is the load factor

- Average time for doing a search will be \( (1 + \alpha) \)
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Odds & Ends

• Hashing first recognized as a technique in the 1950’s
• Comes from English word implying *chop and mix*
• Many different types of hashing for dictionary storage out there. This introduction only scratch d the surface

• A *Cryptographic Hash Function* is a hash function that is *almost* impossible to invert efficiently, i.e, given $h(x)$ very difficult to find $x$
  – Almost by necessity requires that function $h$ distributes keys pretty “randomly” over $0, 1, 2, \ldots, m$. If not true, then would have first step towards guessing value of $x$ that produces $h(x)$.
  
  – Example: Password protection. System password file only stores $h(\text{password})$ and not the password itself.
    
    * When user logs in and types password $p$, system checks $h(p)$ against file.
    * If an attacker steals the file it wouldn’t be helpful, since attacker can’t invert hashed password to get original one