Topological Sort

Version of October 11, 2014
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Clearly, for the system not to hang, the graph must be acyclic. It must be a directed acyclic graph (or DAG).
Course dependence chart 09/10

Red: COMP/CSIE Core
Green: COMP/CSIE Required
Purple: CSIE (NW) Required
Blue: CSIE (MC) Required

Course offering schedule shown here is for reference only; the actual offering schedule may vary slightly from year to year.

F and S means offered in Fall and Spring respectively.

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E.g., order in which classes can be taken:

![Diagram of a directed graph](image)

Topological ordering may not be unique as there are many “equal” elements!

E.G., there are several topological orderings:

- 0, 6, 1, 4, 3, 2, 5, 7, 8, 9
- 0, 4, 1, 6, 2, 5, 3, 7, 8, 9
- ...
Observations

- A DAG must contain at least one vertex with in-degree zero (why?)
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Algorithm: Topological Sort

1. Output a vertex $u$ with in-degree zero in current graph.
2. Remove $u$ and all edges $(u, v)$ from current graph.
3. If graph is not empty, goto step 1.

Correctness

At every stage, current graph is a DAG (why?)

Because current graph is always a DAG, algorithm can always output some vertex. So algorithm outputs all vertices.

Suppose order output was not a topological order. Then there is some edge $(u, v)$ such that $v$ appears before $u$ in the order.

This is impossible, though, because $v$ can not be output until edge $(u, v)$ is removed!
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Topological Sort Algorithm

Topological_sort(G)

Initialize $Q$ to be an empty queue;

\textbf{foreach} $u$ in $V$ \textbf{do}

\hspace{1em} \textbf{if} in-degree($u$) = 0 \textbf{then}

\hspace{2em} // Find all starting vertices

\hspace{2em} Enqueue($Q$, $u$);

\hspace{1em} \textbf{end}

\textbf{end}

\textbf{while} $Q$ is not empty \textbf{do}

\hspace{1em} $u$ = Dequeue($Q$);

\hspace{1em} Output $u$;

\hspace{1em} \textbf{foreach} $v$ in Adj($u$) \textbf{do}

\hspace{2em} // remove $u$’s outgoing edges

\hspace{2em} in-degree($v$) = in-degree($v$) − 1;

\hspace{2em} \textbf{if} in-degree($v$) = 0 \textbf{then}

\hspace{3em} Enqueue($Q$, $v$);

\hspace{2em} \textbf{end}

\hspace{1em} \textbf{end}

\textbf{end}
Example

$Q = \{\}$

$Q = \{0\}$
Example

\[ Q = \{6, 1, 4\} \]

Output: 0

\[ Q = \{1, 4, 3\} \]

Output: 0, 6
Example

\( Q = \{4, 3, 2\} \)

Output: 0, 6, 1

\( Q = \{3, 2\} \)

Output: 0, 6, 1, 4
Example

\[ Q = \{2\} \]
Output: 0, 6, 1, 4, 3

\[ Q = \{7, 5\} \]
Output: 0, 6, 1, 4, 3, 2
Example

\[ Q = \{5, 8\} \]

Output: 0, 6, 1, 4, 3, 2, 7

\[ Q = \{8\} \]

Output: 0, 6, 1, 4, 3, 2, 7, 5
Example

$Q = \{9\}$

Output: 0, 6, 1, 4, 3, 2, 7, 5, 8

$Q = \{\}$

Output: 0, 6, 1, 4, 3, 2, 7, 5, 8, 9

Done!
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For each vertex, we examine all outgoing edges.

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Therefore, the running time is \( O(V + E) \)

Question

Can we use DFS to implement topological sort?