Quick Review of Linearity of Expectation
Linearity of Expectation is one of the simplest and most useful tools used in the analysis of randomized algorithms.

In its easiest form it just says that, if $X, Y$ are any two random variables (not necessarily independent) then

$$E(X + Y) = E(X) + E(Y).$$

The iterated version is that if $X_1, X_2, \ldots, X_n$ are any random variables, then

$$E\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} E(X_i).$$
Example: Let $Z$ be the value seen when rolling two dice. $Z = X_1 + X_2$ where $X_i$ is the value seen when rolling single die $i = 1, 2$. It’s easy to calculate that

$$E(X_i) = \sum_{j=1}^{6} j \Pr(X_i = j) = \sum_{j=1}^{6} j \frac{1}{6} = \frac{7}{2}.$$  

Then

$$E(Z) = E(X_1 + X_2) = E(X_1) + E(X_2) = \frac{7}{2} + \frac{7}{2} = 7.$$
When flipping \( n \) coins, what is the expected number of heads?

\[
Z = \sum_{i=1}^{n} X_i, \text{ where}
\]

\( X_i = 1 \) if coin \( i \) is a head and \( 0 \) if it is a tail.

Set \( \Pr(X_i = 1) = p_i \) and \( \Pr(X_i = 0) = 1 - p_i \).

Then \( X_i \) is a Bernoulli Random Variable with probability \( p_i \).

Note that \( E(X_i) = 1 \cdot \Pr(X_i = 1) = p_i \) so

\[
E(Z) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \Pr(X_i = 1) = \sum_{i=1}^{n} p_i.
\]

Examples:

\( p_i = p \) (all coins the same) \( \Rightarrow E(Z) = pn \)

\( p_i = \frac{1}{i} \) \( \Rightarrow E(Z) = \sum_{i=1}^{n} p_i = \sum_{i=1}^{n} \frac{1}{n} = H_n \sim \ln n \)
Suppose you are flipping $n$ coins, each with $p_i = \frac{1}{2}$, i.e., fair coins. How many times does the pattern $HHH$ appear?

Let $x_1, x_2, \ldots, x_n$ be the list of coin tosses, i.e., $x_i \in \{\text{H(ead)}, \text{T(ail)}\}$.

$$Z = \sum_{i=3}^{n} X_i \text{ where } X_i = 1 \text{ iff } x_{i-2}x_{i-1}x_i = HHH$$

$\Pr(X_i = 1) = \frac{1}{8}$, so

$$E(Z) = \frac{n - 2}{8}.$$
Suppose an algorithm’s input is a permutation of \( n \) numbers. Let \( x_1, x_2, \ldots, x_n \) be the input in its given order.

\( x_i \) is a left to right maxima if it’s bigger than \( x_1, x_2, \ldots, x_{i-1} \).

For example, the red items in these two permutations are the l.t.r. maxima:

\[
5 \ 4 \ 7 \ 8 \ 1 \ 6 \ 3 \ 2 \quad 1 \ 3 \ 5 \ 7 \ 2 \ 4 \ 6 \ 8
\]

Some algorithms’ run times depend upon \( Z \), the number of l.t.r. maxima. Assuming all \( n! \) permutations are equally likely, how can we find \( E(Z) \)?

\[
Z = \sum_{i=1}^{n} X_i \text{ where } X_i = 1 \text{ iff } x_i \text{ is a l.t.r. maxima and } 0 \text{ otherwise}
\]

One way of generating a random permutation is to first randomly choose the first \( i \) items equally likely among all possible \( \binom{n}{i} \) subsets. Then choose a random permutation among the \( i \) possible permutations to order them as \( x_1, \ldots x_i \). Then randomly order the remaining items as \( x_{i+1}, \ldots x_n \).

Probability \( x_i \) is l.t.r. maxima is prob it’s largest in first \( i \) items which is now \( \frac{1}{i} \). So \( X_i \) is Bernouli Random Variable with \( p_i = 1/i \) and \( E(Z_i) = H_n \).
Suppose we flip \( n \) coins. 

\( i \)th coin is Heads with probability \( p_i = 1/i \).

If \( i \)th coin is Heads you get \( i \) dollars; Tails, you get nothing. What is expected amount you receive?

Let \( Z \) be total amount. \( Z = \sum_{i=1}^{n} X_i \) where \( X_i = i \) if \( i \)th coin is heads and is otherwise 0. Then

\[
E(X_i) = ip_i = 1
\]

So,

\[
E(Z) = E\left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} 1 = n
\]
Make a minor change from the previous page.

Suppose we flip $n$ coins. $i$th coin is Heads with probability $p_i = 1/i$.

If $i$th coin is Heads you run another random process $Y_i$ to tell you how much money you receive. All you know is that $E(Y_i) = i$. If $i$th coin is Tails, you get nothing. What is expected amount you receive?

Let $Z$ be total amount. $Z = \sum_{i=1}^{n} X_i Y_i$ where $X_i = 1$ if $i$th coin is Heads and is otherwise 0, so $E(X_i) = p_i$. $E(Y_i) = i$.

Then, because $X_i$ and $Y_i$ are independent

$$E(X_i Y_i) = E(X_i) E(Y_i) = \frac{1}{i} i = 1$$

So,

$$E(Z) = E \left( \sum_{i=1}^{n} X_i Y_i \right) = \sum_{i=1}^{n} E(X_i Y_i) = \sum_{i=1}^{n} 1 = n$$