Dynamic Programming: The Rod Cutting Problem

Version of November 5, 2014
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1. Analyze the structure of an optimal solution
2. Recursively define the value of an optimal solution
3. Compute the value of an optimal solution (usually bottom-up)
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- In D, (usually) work bottom-up. Solve all smaller size problems and build larger problem solutions from them.
  - In DP, many large subproblems reuse solution to same smaller problem.
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**Main idea of DP**

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Input: We are given a rod of length $n$ and a table of prices $p_i$ for $i = 1, \ldots, n$; $p_i$ is the price of a rod of length $i$. 

Example: $n = 4$ and $p_1 = 1$, $p_2 = 5$, $p_3 = 8$, $p_4 = 9$

- If we do not cut the rod, we can earn $p_4 = 9$
- If we cut it into 4 pieces of length 1, we earn $4 \cdot p_1 = 4$
- If we cut it into 2 pieces of length 1 & a piece of length 2, we earn $2 \cdot p_1 + p_2 = 11$
- If we cut it into 2 pieces of length 2, we can earn $2 \cdot p_2 = 10$

There are more options, but the maximum revenue is 10.

In general, rod of length $n$ can be cut in $2^{n-1}$ different ways, since we can choose cutting, or not cutting, at all distances $i$ ($1 \leq i \leq n-1$) from the left end.
Rod Cutting

- **Input**: We are given a rod of length \( n \) and a table of prices \( p_i \) for \( i = 1, \ldots, n \); \( p_i \) is the price of a rod of length \( i \).

- **Goal**: to determine the maximum revenue \( r_n \), obtainable by cutting up the rod and selling the pieces.
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  - There are more options, but the maximum revenue is \( 10 \)

- In general, rod of length \( n \) can be cut in \( 2^{n-1} \) different ways, since we can choose cutting, or not cutting, at all distances \( i \) (\( 1 \leq i \leq n - 1 \)) from the left end.
We can calculate the maximum revenue \( r_n \) in terms of optimal revenues for shorter rods

\[
    r_n = \max(p_n, r_1 + r_{n-1}, r_2 + r_{n-2}, \ldots, r_{n-1} + r_1)
\]

- \( p_n \) if we do not cut at all
- \( r_1 + r_{n-1} \) if we take the sum of optimal revenues for 1 and \( n-1 \)
- \( r_2 + r_{n-2} \) if we take the sum of optimal revenues for 2 and \( n-2 \)
- \( \ldots \)
Optimal Solution

- We can calculate the maximum revenue \( r_n \) in terms of optimal revenues for shorter rods

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  - \( \ldots \)

- Another approach. Set \( r_0 = 0 \) and

\[
r_n = \max_{1 \leq i \leq n} (p_i + r_{n-i})
\]

- Cut a piece of length \( i \), with remainder of length \( n - i \)
- Only the remainder, and not the first piece, may be further divided
Recursive Top-down Implementation

Cut-Rod\((p, n)\)

\[
\begin{aligned}
\text{if } n = 0 \text{ then} \\
\quad \text{return } 0;
\end{aligned}
\]

\[
\begin{aligned}
\text{end} \\
q = -\infty;
\end{aligned}
\]

\[
\begin{aligned}
\text{for } i = 1 \text{ to } n \text{ do} \\
\quad q = \max(q, p[i] + \text{Cut-Rod}(p, n - i));
\end{aligned}
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Algorithm Time
Recursive Top-down Implementation

\textbf{Cut-Rod}(p, n)

\begin{algorithm}
\begin{algorithmic}
\If {\(n = 0\)}
    \State \textbf{return} 0;
\EndIf
\State \(q = -\infty\);
\For {\(i = 1\) to \(n\)}
    \State \(q = \max(q, p[i] + \text{Cut-Rod}(p, n - i));\)
\EndFor
\State \textbf{return} \(q\);
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\textbf{Algorithm Time}

- \(T(n)\): the total number of calls made to \text{Cut-Rod} when called with rod length \(n\)
Recursive Top-down Implementation

Cut-Rod($p, n$)

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\begin{align*}
\text{if } n &= 0 \text{ then} \\
&\quad \text{return } 0; \\
\text{end} \\
q &= -\infty; \\
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&\quad q = \max(q, p[i] + \text{Cut-Rod}(p, n - i)); \\
\text{end} \\
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Algorithm Time

- $T(n)$: the total number of calls made to Cut-Rod when called with rod length $n$

\[
T(n) = \begin{cases} 
1 + \sum_{0 \leq j \leq n-1} T(j), & \text{if } n > 0, \\
1, & \text{if } n = 0.
\end{cases}
\]
Recursive Top-down Implementation

Cut-Rod\((p, n)\)

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- Induction \(\Rightarrow T(n) = 2^n\)
Algorithm calls same subproblem many times

\[ p_1 + r_3 \quad p_2 + r_2 \quad p_3 + r_1 \quad p_4 \]
After solving a *subproblem*, store the solution.
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  - Next time you encounter same subproblem, lookup the solution, instead of solving it again
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- Uses space to save time.
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Two main methodologies: top-down and bottom-up
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Two main methodologies: top-down and bottom-up
  - Corresponding algorithms have the same asymptotic cost, but bottom-up is usually faster in practice
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Main idea of bottom-up DP
Concept of DP

- After solving a *subproblem*, store the solution
  - Next time you encounter same subproblem, lookup the solution, instead of solving it again
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- Two main methodologies: top-down and bottom-up
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- Main idea of bottom-up DP
  - Don’t wait until until subproblem is encountered.
  - Sort the subproblems by size; solve smallest subproblems first
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Main idea of bottom-up DP
- Don’t wait until until subproblem is encountered.
- Sort the subproblems by size; solve smallest subproblems first
- Combine solutions of small subproblems to solve larger ones


- \( p_i \) are the problem inputs.
- \( r_i \) is max profit from cutting rod of length \( i \).
- Goal is to calculate \( r_n \)
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- Goal is to calculate $r_n$
- $r_i$ defined by
  - $r_1 = 1$ and $r_n = \max_{1 \leq i \leq n}(p_i + r_{n-i})$
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Iteratively fill in $r_i$ table by calculating $r_1, r_2, r_3, \ldots$

$r_n$ is final solution
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• Iteratively fill in $r_i$ table by calculating $r_1, r_2, r_3, \ldots$
• $r_n$ is final solution

<table>
<thead>
<tr>
<th>i</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>\ldots</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>$r_i$</td>
<td>$p_1$</td>
<td></td>
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<td>\ldots</td>
<td></td>
</tr>
</tbody>
</table>
Bottom-Up-Cut-Rod(p, n)

\[
\begin{align*}
& r[0] = 0; \quad // \text{Array } r[0...n] \text{ stores the computed optimal values} \\
& \text{for } j = 1 \text{ to } n \text{ do} \\
& \quad \quad \quad // \text{Consider problems in increasing order of size} \\
& \quad \quad \quad q = -\infty; \\
& \quad \quad \quad \text{for } i = 1 \text{ to } j \text{ do} \\
& \quad \quad \quad \quad // \text{To solve a problem of size } j, \text{ we need to consider all} \\
& \quad \quad \quad \quad \text{decompositions into } i \text{ and } j - i \\
& \quad \quad \quad \quad q = \max(q, p[i] + r[j - i]); \\
& \quad \quad \quad \text{end} \\
& \quad \quad \quad r[j] = q; \\
& \text{end} \\
& \text{return } r[n];
\end{align*}
\]

Cost: \(O(n^2)\)

The outer loop computes \(r[1], r[2], \ldots, r[n]\) in this order.
To compute \(r[j]\), the inner loop uses all values \(r[0], r[1], \ldots, r[j - 1]\) (i.e., \(r[j - i]\) for \(1 \leq i \leq j\)).
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  - To compute \(r[j]\), the inner loop uses all values \(r[0], r[1], \ldots, r[j - 1]\) (i.e., \(r[j - i]\) for \(1 \leq i \leq j\))
Outputting the Cutting

- Algorithm only computes $r_i$. It does not output the cutting.
- Easy fix
  - When calculating $r_j = \max_{1 \leq i \leq j}(p_i + r_{j-i})$
    - store value of $i$ that achieved this max in new array $s[j]$.
    - This $j$ is the size of last piece in the optimal cutting.
- After algorithm is finished, can reconstruct optimal cutting by unrolling the $s_j$. 
Extended-Bottom-Up-Cut-Rod\((p, n)\)

// Array \(s[0...n]\) stores the optimal size of the first piece to cut off
\(r[0] = 0;\) // Array \(r[0...n]\) stores the computed optimal values
for \(j = 1\) to \(n\) do
    \(q = -\infty;\)
    for \(i = 1\) to \(j\) do
        // Solve problem of size \(j\)
        if \(q < p[i] + r[j - i]\) then
            \(q = p[i] + r[j - i];\)
            \(s[j] = i;\) // Store the size of the first piece
        end
    end
    \(r[j] = q;\)
end
while \(n > 0\) do
    // Print sizes of pieces
    Print \(s[n];\)
    \(n = n - s[n];\)
end