An Introduction to Hashing

(Following CLRS)
Outline

- Introduction
- Hashing with Chaining
- Open Addressing
- Hash Functions & Universal Hashing
- Odds & Ends
Introduction

Known: A set $U = \{1, 2, \ldots, u - 1\}$ of the universe of possible keys that could exist.

Goal: To maintain a dictionary that permits the following operation on keys

- **Search**(x): Find the record with key $x$ or report that it does not exist
- **Insert**(x): Insert a new record with key $x$
- **Delete**(x): Delete the record with key $x$

Would like $O(1)$ (average) time per operation.
Introduction (ii)

Universe Size: $U$
Number of actual keys: $n$ \((n << U)\)

Will store keys in a size $m$ array \((m \text{ “close” to } n)\).
Need a way to assign key $k$ to array location.
Use a hash function $h$

\[
h : U \rightarrow \{0, 1, \ldots, m - 1\}
\]

$\alpha = \frac{n}{m}$ is load factor,
average \# of existing keys with same $h(x)$

For now, assume uniform hashing, that, every key is equally likely to hash to any of the $m$ slots,

\[
\forall x, i, \quad \Pr (h(x) = i) = \frac{1}{m}.
\]
Introduction (iii)

\( h : U \rightarrow \{0, 1 \ldots, m - 1\} \)

\( h \) maps the set of keys into a “small” table. Key \( k \) is stored in table slot \( h(k) \).

Finding key \( k \) is then just a matter of going to table location \( h(k) \).

Problem is that, since \( m \) is small, many keys might be mapped to same slot, creating collision.

Two major approaches to addressing collisions:
(1) Chaining
(2) Open Addressing
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Chaining

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

All elements that hash to the same slot are put into the same linked list.

- **Insert(x):** Insert \( x \) into front of list for slot \( h(x) \) \( O(1) \)
- **Delete(x):** Delete \( x \) from list for slot \( h(x) \), if it's there.
  - Use doubly linked lists \( O(1) \)
- **Search(x):** Search for \( x \) in list for \( h(x) \) \( O(\text{length of list}) \)
Chaining: Unsuccessful Search

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

Search(x): Search for \( x \) in list
for \( h(x) \) \( O(\text{length of list}) \)

Recall load factor \( \alpha = \frac{n}{m} \).

This is average \# items per list.

Unsuccessful search for \( x \) not in table will require searching entire list for \( h(x) \).

Worst case length is \( O(1) \).
Average case length is \( O(\alpha) \).

Average Unsuccessful Search time is \( O(1 + \alpha) \)
where 1 is amount of time to calculate \( h(x) \).
Chaining: Successful Search

For Successful Search for $x$:
Assume $x$ equally likely to be any item in table
Search cost is \# items ahead of $x$ in list $h(x)$
\[
= \# \text{ of items inserted into } h(x) \text{ after } x
\]

If $x$ is $i$'th item inserted
\[
\Rightarrow n - i \text{ items inserted after } x
\Rightarrow \alpha - \frac{i}{m} = \frac{n-i}{m} \text{ items inserted on average into } h(x) \text{ after } x
\]

$x$ is equally likely (with prob $1/n$) to be $i$'th inserted item.

Average \# of items ahead of $x$ in list $h(x)$ is
\[
\frac{1}{n} \sum_{i=1}^{n} \left( \alpha - \frac{i}{m} \right) = \alpha - \frac{n(n+1)}{2nm} = \alpha - \frac{\alpha}{2} + \frac{\alpha}{2n}
\]

Adding 1 unit of time to calculate $h(x)$
Average cost of successful search is $\Theta(1 + \alpha)$. 
Chaining

$h : U \rightarrow \{0,1\ldots,m-1\}$

Search($x$): Search for $x$ in list for $h(x)$ \(O(\text{length of list})\)

Both Successful and Unsuccessful Search require \(O(1 + \alpha)\) time on average

where \(\alpha = \frac{n}{m}\) is the load factor
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Open Addressing

\[ h : U \to \{0, 1 \ldots, m - 1\} \]

- No lists. All keys stored in hash table itself.
- For insertion, \textit{probe} hash table until empty slot for insertion is found.
- \textit{Probe Sequence} is part of hash function.
- Hash function is now
  \[ h : U \times \{0, 1, \ldots, m - 1\} \to \{0, 1, \ldots, m - 1\} \]
- Probe sequence for \( x \) is,
  \[ h(x, 0), h(x, 1), \ldots, h(x, m - 1) \]
  which is a \textit{permutation} of \( \{0, 1, \ldots, m\} \)
- For search(\( x \)), \textit{probe} hash table using probe sequence for \( h(x) \) until either \( x \) or empty slot for insertion is found.
Open Addressing: Linear Probing

$h' : U \rightarrow \{0, 1 \ldots, m - 1\}$

- Hash Function is $h(x, i) = (h'(x) + i) \mod m$ where $h'(x)$ is original hash function.

- **Insert**: Attempts insertion at $h'(x)$, then $h'(x) + 1$, $h'(x) + 2$, etc., (wrapping around to 0 after reaching end of table) until empty slot is found and $x$ inserted there.

- **Search(x)**: Examines probe sequence until it finds $x$ or an empty slot. If empty slot is found then $x$ wasn’t previously inserted and search unsuccessful.

- **Deletion**: More complicated. *Can’t actually delete item and reset slot as ‘empty” That would mess up Search(x). Can mark slot as (used but) deleted. Deletion in open addressing does cause difficulties. Better to use chaining.*
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \]

\[ h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

*Only for illustration. This is a BAD hash function*

Insert(15)
Open Addressing: Linear Probing

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Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad \text{and} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Open Addressing: Linear Probing

\[ h' : U \to \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Search(11) Exists
**Open Addressing: Linear Probing**

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Search(11) Exists

Search(3) Exists

Search(9) Does not exist
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

\begin{align*}
0 & \quad \text{Search}(11) \quad \text{Exists} \\
11 & \quad \text{Search}(3) \quad \text{Exists} \\
15 & \quad \text{Search}(9) \quad \text{Does not exist} \\
3 & \quad \text{Search}(24) \\
18 & \\
35 &
\end{align*}

Only for illustration. This is a BAD hash functionn
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad \text{where} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

*Only for illustration. This is a BAD hash function*

Easy to code but suffers from primary clustering.
Long runs build up, increasing average search time

One fix is to change probe sequence to no longer be linear.
Open Addressing: Quadratic Probing

\[ h' : U \to \{0, 1 \ldots, m - 1\} \]

- Hash Function is \( h(x, i) = (h'(x) + c_1 i + c_2 i^2) \mod m \)
  where \( h'(x) \) is original hash function and \( c_1, c_2 \) fixed constants.

- As example we will set
  \( h'(x) = x \mod 12, \ c_1 = 0 \) and \( c_2 = 1 \) so
  \( h(x, i) = (x + i^2) \mod 12 \)

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Open Addressing: Quadratic Probing

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$h' : U \rightarrow \{0, 1 \ldots, m - 1\} \rightarrow$

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Insert(35)
Open Addressing: Double Hashing

\( h' : U \rightarrow \{0, 1 \ldots, m - 1\} \)

- Hash Function is \( h(x, i) = (h_1(x) + ih_2(x)) \mod m \)
- \( h_1(x) \) and \( h_2(x) \) are auxiliary hash functions
- Note that (unlike before) probe sequence depends upon \( x \)
- In order for probe sequence to check entire table, must have \( h_2(x) \) be relatively prime to \( m \), e.g.,
  - \( m \) a power of 2; \( h_2(x) \) always odd
  - \( m \) prime; \( h_2(x) \) always less than \( m \).

Example: \( m = 13 \)
\[
\begin{align*}
  h_1(x) &= x \mod m \\
  h_2(x) &= 1 + (x \mod 11)
\end{align*}
\]

14 would have probe sequence 1, 5, 9, \ldots
Since first 2 locations full, it will be inserted into 9.
Open Addressing: Analysis Results

We have seen 3 different open addressing collision resolution methods:

- **Linear Probing**: \( h(x, i) = (h'(x) + 1) \mod m \)
- **Quadratic Probing**: \( h(x, i) = (h'(x) + c_1 + c_2x^2) \mod m \)
- **Double Hashing**: \( h(x, i) = (h_1(x) + ih_2(x)) \mod m \)

For analysis, we often assume uniform hashing. This states that the probe sequence
\[ h(x, 0), h(x, 1), h(x, 2), \ldots, h(x, m) \]
is equally likely to be any of the \( m! \) permutations of \( 1, 2, \ldots, m \).

Uniform Hashing is not actually realizable. The more random our probe sequence, though, the closer actual behavior is to theory.
Open Addressing: Analysis Results

Recall $\alpha = \frac{n}{m}$ is the load factor.
In what follows we assume uniform hashing.

**Lemma:** Given an open-address hash table with load factor $\alpha = \frac{n}{m} < 1$, the average number of probes in an unsuccessful search is at most

$$\frac{1}{1 - \alpha}$$

**Lemma:** Inserting an element into an open-address hash table with load factor $\alpha = \frac{n}{m} < 1$, requires, on average, at most $\frac{1}{1 - \alpha}$ probes.

**Lemma:** Given an open-address hash table with load factor $\alpha = \frac{n}{m} < 1$, the average number of probes in a successful search is at most

$$\frac{1}{\alpha \ln \frac{1}{1 - \alpha}}$$
Outline

• Introduction
• Hashing with Chaining
• Open Addressing
• Hash Functions & Universal Hashing
• Odds & Ends
Hash Functions & Universal Hashing

Returning to chained hashing, notice that our analysis assumed that the hashed keys were equally distributed among the slots.

- If all keys hashed to same slot, performance would be very bad.
- If the hash function $h(x)$ is given in advance and $n << \mathcal{U}$, very easy to construct bad case in which all keys map to the same slot.

- We sidestep this issue by choosing a random hash function.
- More specifically, we will have a collection of hash functions $\mathcal{H}$.
- Given any set of keys, we will choose a random hash function $h \in \mathcal{H}$ and then hash using $h(x)$.

- On average, the set of $n$ keys will be hashed so that each slot will get $O(n/m) = O(\alpha)$ keys.

- Our $O(1 + \alpha)$ successful/unsuccessful search times for chained hashing will then hold on average.

- One class $\mathcal{H}$ of hash functions having this property are the Universal ones; they permit Universal Hashing.
Universal Hashing

- Let \( \mathcal{H} \) be a set of hash functions, such that each \( h \in \mathcal{H} \) maps
  \[
  h : U \rightarrow \{0, 1, \ldots, m - 1\}
  \]
- \( \mathcal{H} \) is \textbf{Universal} if, for every two different keys \( k, \ell \), the number of hash functions in \( \mathcal{H} \) that map \( k, \ell \) to the same slot is at most \( |\mathcal{H}|/m \), i.e.,
  \[
  \forall k \neq \ell \in U, \quad |\{h \in \mathcal{H} : h(k) = h(\ell)\}| \leq \frac{|\mathcal{H}|}{m}.
  \]

Let \( k_1, k_2, \ldots, k_n \) be the \( n \) keys. Let \( i \) be any fixed index.
Then, for \( j \neq i \), if \( h \in \mathcal{H} \) is chosen uniformly at random,
\[
\Pr(h(k_i) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{m} = \frac{1}{m}.
\]
From linearity of expectation, if \( h \in \mathcal{H} \) is chosen uniformly at random, average \# of other keys mapping to the same slot as \( k_i \) is then
\[
\sum_{j \neq i; 1 \leq j \leq n} \Pr(h(k_i) = h(k_j)) \leq \frac{n - 1}{m} < \alpha
\]
Similarly, if \( k \) is not one of the \( n \) keys then, for all \( j \),
\[
\Pr(h(k) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{m} = \frac{1}{m}
\]
and average \# of keys mapping to same slot as \( k \) is
\[
\sum_{j=1}^{n} \Pr(h(k) = h(k_j)) \leq \frac{n}{m} = \alpha
\]
Construction of Universal Hash Functions

- Choose prime $p > U$
- Set $Z_p^* = \{1, 2, 3, \ldots, p - 1\}$ and $Z_p = \{0, 1, 2, 3, \ldots, p - 1\}$
- Define
  \[
  \forall a \in Z_p^*, \ b \in Z_p, \ h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m
  \]

Example: Set $p = 17$, $m = 6$. Then

\[
  h_{3,4}(8) = ((3 \cdot 8 + 4) \mod 17) \mod 6 = 5
  \]

Lemma: The Class $\mathcal{H} = \{ h_{a,b} : a \in Z_p^*, \ b \in Z_p \}$ is Universal.

Proof: Need to show that for all $k \neq \ell$, number of pairs $(a, b)$ with $h_{a,b}(k) = h_{a,b}(\ell)$ is $\leq p(p - 1)/m$
Construction of Universal Hash Functions (ii)

\( \forall a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p, \quad h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m \)

\( p \) prime, \( \mathbb{Z}_p^* = \{1, 2, 3, \ldots, p - 1\}, \quad \mathbb{Z}_p = \{0, 1, 2, 3, \ldots, p - 1\} \)

(1) Let \( k \neq \ell \in U \). For given \((a, b) \in \mathbb{Z}_p^* \times \mathbb{Z}_p\) set
\[
r = (ak + b) \mod p, \quad s = (a\ell + b) \mod p
\]

(2) Every different \((a, b)\) pair generates a unique \((r, s)\) pair. This is because for a given \((r, s)\) pair we can (uniquely) solve
\[
a = (r - s)(k - \ell)^{-1} \mod p, \quad b = (r - ak) \mod p.
\]

where \((k - \ell)^{-1}\) is the multiplicative inverse base \( p \). Since, for fixed \( p, k, \ell \), we must have \( r \neq s \), there are \( p(p - 1) \) \((r, s)\) pairs. Since there are also \( p(p - 1) \) \((a, b)\) pairs, there is a one-one correspondence between them, with every \((a, b)\) pair generating a different \((r, s)\).
Construction of Universal Hash Functions (iii)

∀a ∈ Z_p^*, b ∈ Z_p, \ h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m

p prime, \ Z^*_p = \{1, 2, 3, \ldots, p - 1\}, \ Z_p = \{0, 1, 2, 3, \ldots, p - 1\}

k \neq \ell. \ (a, b) ∈ Z^*_p \times Z_p. \ r = (ak + b) \mod p, \ s = (a\ell + b) \mod p

(2) Every different (a, b) pair generates a unique (r, s) pair, r \neq s.

(3) # of (a, b) pairs for which \ h_{a,b}(k) = h_{a,b}(\ell) \ is \ \leq p(p - 1)/m.

\ h_{a,b}(k) = h_{a,b}(\ell) \ \iff \ r \equiv s \mod m.

For fixed r, # of s \neq r with r \equiv s \mod m \ is
\leq \lceil p/m \rceil - 1 \leq (p - 1)/m

Summing over all p possible values of r gives \leq p(p - 1)/m pairs (r, s) with s \neq r and r \equiv s \mod m,
i.e., \leq p(p - 1)/m = |H|/m pairs (a, b) with \ h_{a,b}(k) = h_{a,b}(\ell)

⇒ H is Universal
Universal Hashing: Wrap Up

• Just saw that the set of Hash functions

\[ \mathcal{H} = \{ h_{a,b} : a \in \mathbb{Z}_p^*, b \in \mathbb{Z}_p \} \]

is Universal

• This implies that for any set of \( n \) keys \( K = \{ k_1, k_2, \ldots, k_n \} \), an effective way of storing the keys is to
  – Choose a random pair \((a, b)\) uniformly at random from the \( p(p - 1) \) pairs in \( \mathbb{Z}_p^* \times \mathbb{Z}_p \)
  – Hash the items in \( K \) using hash function \( h_{a,b} \)

• Because \( \mathcal{H} \) is Universal, average time for storing the data will be \( O(n\alpha) \) where \( \alpha = n/m \) is the load factor

• Average time for doing a search will be \( (1 + \alpha) \)
Outline

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• Hashing first recognized as a technique in the 1950’s
• Comes from English word implying *chop and mix*
• Many different types of hashing for dictionary storage out there. This introduction only scratched the surface.

• A *Cryptographic Hash Function* is a hash function that is *almost* impossible to invert efficiently, i.e., given $h(x)$ very difficult to find $x$.
  – Almost by necessity requires that function $h$ distributes keys pretty “randomly” over $0, 1, 2, \ldots, m$. If not true, then would have first step towards guessing value of $x$ that produces $h(x)$.
  – Example: Password protection. System password file only stores $h(password)$ and not the password itself.
    * When user logs in and types password $p$, system checks $h(p)$ against file.
    * If an attacker steals the file it wouldn’t be helpful, since attacker can’t invert hashed password to get original one.