An Introduction to Hashing

*(Following CLRS)*
Outline

• Introduction
• Hashing with Chaining
• Open Addressing
• Hash Functions & Universal Hashing
• Odds & Ends
Known: A set $U = \{1, 2, \ldots, u - 1\}$ of the universe of possible keys that could exist.

Goal: To maintain a dictionary that permits the following operation on keys

- **Search(x):** Find the record with key $x$ or report that it does not exist
- **Insert(x):** Insert a new record with key $x$
- **Delete(x):** Delete the record with key $x$
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Would like $O(1)$ (average) time per operation.
Introduction (ii)

Universe Size: $U$

Number of actual keys: $n$ ($n << U$)
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Use a hash function $h$

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Need a way to assign key $k$ to array location.
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 average \# of existing keys with same $h(x)$
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$\alpha = \frac{n}{m}$ is load factor,
average # of existing keys with same $h(x)$

For now, assume uniform hashing, that, every key is equally likely to hash to any of the $m$ slots,

$$\forall x, i, \quad \Pr (h(x) = i) = \frac{1}{m}.$$
Introduction (iii)

\[ h : U \to \{0, 1 \ldots, m - 1\} \]

\( h \) maps the set of keys into a “small” table. Key \( k \) is stored in table slot \( h(k) \).

Finding key \( k \) is then just a matter of going to table location \( h(k) \).

Problem is that, since \( m \) is small, many keys might be mapped to same slot, creating collision.
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Problem is that, since \( m \) is small, many keys might be mapped to same slot, creating \textit{collision}.

Two major approaches to addressing collisions:
(1) Chaining
(2) Open Addressing
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Chaining

$h : U \rightarrow \{0, 1 \ldots, m - 1\}$

All elements that hash to the same slot are put into the same linked list.
Chaining

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Insert($x$): Insert $x$ into front of list for slot $h(x)$

Delete($x$): Delete $x$ from list for slot $h(x)$, if it’s there.

Use doubly linked lists

Search($x$): Search for $x$ in list for $h(x)$
Chaining

$$h: U \rightarrow \{0, 1 \ldots, m - 1\}$$

All elements that hash to the same slot are put into the same linked list

**Insert(x):** Insert $x$ into front of list for slot $h(x)$ \(O(1)\)

**Delete(x):** Delete $x$ from list for slot $h(x)$, if it’s there.  
*Use doubly linked lists* \(O(1)\)

**Search(x):** Search for $x$ in list for $h(x)$ \(O(\text{length of list})\)
Chaining: Unsuccessful Search

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**Search**($x$): Search for $x$ in list
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Chaining: Unsuccessful Search

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

**Search(x):** Search for \( x \) in list for \( h(x) \) \( O(\text{length of list}) \)

Recall load factor \( \alpha = \frac{n}{m} \).

This is average \# items per list.

Unsuccessful search for \( x \) not in table will require searching entire list for \( h(x) \).

Worst case length is \( O(1) \).

Average case length is \( O(\alpha) \).
Chaining: Unsuccessful Search

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This is average \# items per list.

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Worst case length is \(O(1)\).

Average case length is \(O(\alpha)\).

Average Unsuccessful Search time is \(O(1 + \alpha)\)
where 1 is amount of time to calculate \(h(x)\).
Chaining: Successful Search

For Successful Search for $x$:
Assume $x$ equally likely to be any item in table
Chaining: Successful Search

For Successful Search for \( x \):
Assume \( x \) equally likely to be any item in table
Search cost is \( \# \) items ahead of \( x \) in list \( h(x) \)
\[
= \# \text{ of items inserted into } h(x) \text{ after } x
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If $x$ is $i$'th item inserted
\[ \Rightarrow n - i \text{ items inserted after } x \]
\[ \Rightarrow \alpha - \frac{i}{m} = \frac{n-i}{m} \text{ items inserted on average into } h(x) \text{ after } x \]

$x$ is equally likely (with prob $1/n$) to be $i$'th inserted item.

Average \# of items ahead of $x$ in list $h(x)$ is

\[
\frac{1}{n} \sum_{i=1}^{n} \left( \alpha - \frac{i}{m} \right) = \alpha - \frac{n(n+1)}{2nm} = \alpha - \frac{\alpha}{2} + \frac{\alpha}{2n}
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Adding 1 unit of time to calculate $h(x)$
Average cost of successful search is $\Theta(1 + \alpha)$.
Chaining

\[ h : U \rightarrow \{0, 1 \ldots, m - 1\} \]

Search(x): Search for \( x \) in list for \( h(x) \) \( O(\text{length of list}) \)

Both Successful and Unsuccessful Search require \( O(1 + \alpha) \) time on average

where \( \alpha = \frac{n}{m} \) is the load factor
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Open Addressing

\( h : U \rightarrow \{0, 1 \ldots, m - 1\} \)

- No lists. All keys stored in hash table itself.
- For insertion, \textit{probe} hash table until empty slot for insertion is found.
- \textit{Probe Sequence} is part of hash function.
- Hash function is now

\[
   h : U \times \{0, 1, \ldots, m - 1\} \rightarrow \{0, 1, \ldots, m - 1\}
\]

- Probe sequence for \( x \) is,

\[
   h(x, 0), h(x, 1), \ldots, h(x, m - 1)
\]

which is a \textit{permutation} of \( \{0, 1, \ldots, m\} \)

- For search(\( x \)), \textit{probe} hash table using probe sequence for \( h(x) \) until either \( x \) or empty slot for insertion is found.
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \]

- Hash Function is \( h(x, i) = (h'(x) + i) \mod m \) where \( h'(x) \) is original hash function.

- **Insert**: Attempts insertion at \( h'(x) \), then \( h'(x) + 1 \), \( h'(x) + 2 \), etc., (wrapping around to 0 after reaching end of table) until empty slot is found and \( x \) inserted there.

- **Search(\( x \))**: Examines probe sequence until it finds \( x \) or an empty slot.
  If empty slot is found then \( x \) wasn’t previously inserted and search unsuccessful

- **Deletion**: More complicated.
Open Addressing: Linear Probing

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- **Search\((x)\)**: Examines probe sequence until it finds \( x \) or an empty slot.
  If empty slot is found then \( x \) wasn’t previously inserted and search unsuccessful

- **Deletion**: More complicated.
  *Can’t actually delete item and reset slot as ‘empty’*
  *That would mess up Search\((x)\).*
  *Can mark slot as (used but) deleted.*
  *Deletion in open addressing does cause difficulties.*
  *Better to use chaining.*
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).
Open Addressing: Linear Probing

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*Only for illustration. This is a BAD hash function*
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Insert(15)
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- Insert(0)
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- Insert(15)
- Insert(0)

\[ \begin{array}{c|c}
0 & 0 \\
1 & \\
15 & \\
\vdots & \\
11 & \\
\end{array} \]
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</table>
Open Addressing: Linear Probing

$h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m$

As example, let $h(x) = x \mod m$ with $m = 12$.

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Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad \text{and} \quad h(x) = (h'(x) + 1) \mod m \]

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Search(11)
Open Addressing: Linear Probing

$h' : U \rightarrow \{0, 1 \ldots, m-1\} \quad h(x) = (h'(x) + 1) \mod m$

As example, let $h(x) = x \mod m$ with $m = 12$.  

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Search(11) Exists
Open Addressing: Linear Probing

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Search(11)   Exists

Search(3)
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

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Search(11) Exists

Search(3)
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Open Addressing: Linear Probing

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Search(11) \hspace{1cm} Exists

Search(3) \hspace{1cm} Exists

Search(9) \hspace{1cm} Does not exist
Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad \text{where} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

*Only for illustration. This is a BAD hash function*

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**Open Addressing: Linear Probing**

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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- Search(11)  Exists
- Search(3)  Exists
- Search(9)  Does not exist
- Search(24)  Does not exist
### Open Addressing: Linear Probing

$h' : U \rightarrow \{0, 1 \ldots, m - 1\}$ \hspace{1cm} $h(x) = \left( h'(x) + 1 \right) \mod m$

As example, let $h(x) = x \mod m$ with $m = 12$.

**Only for illustration. This is a BAD hash function**

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- Search(11) Exists
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\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Open Addressing: Linear Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \quad \quad h(x) = (h'(x) + 1) \mod m \]

As example, let \( h(x) = x \mod m \) with \( m = 12 \).

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Easy to code but suffers from primary clustering. Long runs build up, increasing average search time.
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Easy to code but suffers from primary clustering. Long runs build up, increasing average search time.

One fix is to change probe sequence to no longer be linear.
Open Addressing: Quadratic Probing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \]

- Hash Function is \( h(x, i) = (h'(x) + c_1 i + c_2 i^2) \mod m \)
  where \( h'(x) \) is original hash function and \( c_1, c_2 \) fixed constants.

- As example we will set \( h'(x) = x \mod 12, c_1 = 0 \) and \( c_2 = 1 \) so
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Open Addressing: Quadratic Probing

\( h' : U \to \{0, 1 \ldots, m - 1\} \)

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Open Addressing: Quadratic Probing

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Open Addressing: Quadratic Probing

$h' : U \rightarrow \{0, 1 \ldots, m - 1\}$

- Hash Function is $h(x, i) = (h'(x) + c_1 i + c_2 i^2) \mod m$ where $h'(x)$ is original hash function and $c_1, c_2$ fixed constants.

- As example we will set $h'(x) = x \mod 12, c_1 = 0$ and $c_2 = 1$ so $h(x, i) = (x + i^2) \mod 12$

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Open Addressing: Double Hashing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \]

- Hash Function is \( h(x, i) = (h_1(x) + ih_2(x)) \mod m \)
- \( h_1(x) \) and \( h_2(x) \) are **auxiliary hash functions**
- Note that (unlike before) probe sequence depends upon \( x \)
- In order for probe sequence to check entire table, must have \( h_2(x) \) be relatively prime to \( m \), e.g.,
  - \( m \) a power of 2; \( h_2(x) \) always odd
  - \( m \) prime; \( h_2(x) \) always less than \( m \).
Open Addressing: Double Hashing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \]

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Example: \( m = 13 \)

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\begin{align*}
  h_1(x) &= x \mod m \\
  h_2(x) &= 1 + (x \mod 11)
\end{align*}
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Open Addressing: Double Hashing

\[ h' : U \rightarrow \{0, 1 \ldots, m - 1\} \]

- Hash Function is \( h(x, i) = (h_1(x) + ih_2(x)) \mod m \)
- \( h_1(x) \) and \( h_2(x) \) are auxiliary hash functions
- Note that (unlike before) probe sequence depends upon \( x \)
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<table>
<thead>
<tr>
<th>( h_1(x) )</th>
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<tbody>
<tr>
<td>79</td>
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<tr>
<td>69</td>
<td></td>
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<tr>
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14 would have probe sequence 1, 5, 9, \ldots
Since first 2 locations full, it will be inserted into 9.
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Open Addressing: Analysis Results

We have seen 3 different open addressing collision resolution methods:

- Linear Probing: \[ h(x, i) = (h'(x) + 1) \mod m \]
- Quadratic Probing: \[ h(x, i) = (h'(x) + c_1 i + c_2 x^2) \mod m \]
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For analysis, we often assume **uniform hashing**. This states that the probe sequence

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h(x, 0), h(x, 1), h(x, 2), \ldots, h(x, m)
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Uniform Hashing is not actually realizable. The more random our probe sequence, though, the closer actual behavior is to theory.
Open Addressing: Analysis Results

Recall $\alpha = \frac{n}{m}$ is the load factor.
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Outline

- Introduction
- Hashing with Chaining
- Open Addressing
- Hash Functions & Universal Hashing
- Odds & Ends
Hash Functions & Universal Hashing

Returning to chained hashing, notice that our analysis assumed that the hashed keys were equally distributed among the slots.

- If all keys hashed to same slot, performance would be very bad.
- If the hash function \( h(x) \) is given in advance and \( n << U \), very easy to construct bad case in which all keys map to the same slot.
- We sidestep this issue by choosing a random hash function.
- More specifically, we will have a collection of hash functions \( \mathcal{H} \).
- Given any set of keys, we will choose a random hash function \( h \in \mathcal{H} \) and then hash using \( h(x) \).
- On average, the set of \( n \) keys will be hashed so that each slot will get \( O(n/m) = O(\alpha) \) keys.
- Our \( O(1 + \alpha) \) successful/unsuccessful search times for chained hashing will then hold on average.
- One class \( \mathcal{H} \) of hash functions having this property are the Universal ones; they permit Universal Hashing.
Universal Hashing

- Let $\mathcal{H}$ be a set of hash functions, such that each $h \in \mathcal{H}$ maps $h : U \rightarrow \{0, 1, \ldots, m - 1\}$

- $\mathcal{H}$ is Universal if, for every two different keys $k, \ell$, the number of hash functions in $\mathcal{H}$ that map $k, \ell$ to the same slot is at most $|\mathcal{H}|/m$, i.e.,

$$\forall k \neq \ell \in U, \quad |\{h \in \mathcal{H} : h(k) = h(\ell)\}| \leq \frac{|\mathcal{H}|}{m}.$$
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Let $k_1, k_2, \ldots, k_n$ be the $n$ keys. Let $i$ be any fixed index. Then, for $j \neq i$, if $h \in \mathcal{H}$ is chosen uniformly at random,

$$\Pr(h(k_i) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{m} = \frac{1}{m}.$$

From linearity of expectation, if $h \in \mathcal{H}$ is chosen uniformly at random, average # of other keys mapping to the same slot as $k_i$ is then

$$\sum_{j \neq i; 1 \leq j \leq n} \Pr(h(k_i) = h(k_j)) \leq \frac{n - 1}{m} < \alpha$$
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Similarly, if $k$ is not one of the $n$ keys then, for all $j$,

$$\Pr(h(k) = h(k_j)) \leq \frac{1}{|\mathcal{H}|} \frac{|\mathcal{H}|}{m} = \frac{1}{m} \quad \text{and average # of keys mapping to same slot as } k \text{ is } \sum_{j=1}^{n} \Pr(h(k) = h(k_j)) \leq \frac{n}{m} = \alpha.$$. 


Construction of Universal Hash Functions

- Choose prime $p > U$
- Set $Z_p^* = \{1, 2, 3, \ldots , p - 1\}$ and $Z_p = \{0, 1, 2, 3, \ldots , p - 1\}$
- Define

$$\forall a \in Z_p^*, \ b \in Z_p, \ h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m$$
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Example: Set $p = 17, m = 6$. Then

$$h_{3,4}(8) = \left( (3 \cdot 8 + 4) \mod 17 \right) \mod 6 = 5$$
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Proof: Need to show that for all $k \neq \ell$, number of pairs $(a,b)$ with $h_{a,b}(k) = h_{a,b}(\ell)$ is $\leq \frac{p(p - 1)}{m}$
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\[ \forall a \in Z^*_p, \ b \in Z_p, \quad h_{a,b}(x) = \left( (ax + b) \mod p \right) \mod m \]

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(1) Let \( k \neq \ell \in U \). For given \((a, b) \in Z_p^* \times Z_p\) set

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(2) Every different (a, b) pair generates a unique (r, s) pair

This is because for a given (r, s) pair we can (uniquely) solve

\[ a = (r - s)(k - \ell)^{-1} \mod p, \quad b = (r - ak) \mod p. \]

where \((k - \ell)^{-1}\) is the multiplicative inverse base p. Since, for fixed \(p, k, \ell\), we must have \(r ≠ s\), there are are \(p(p - 1)\) (r, s) pairs. Since there are also \(p(p - 1)\) (a, b) pairs, there is a one-one correspondence between them, with every (a, b) pair generating a different (r, s).
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\[ \Rightarrow \mathcal{H} \text{ is Universal} \]
Universal Hashing: Wrap Up

• Just saw that the set of Hash functions
  \[ H = \{ h_{a,b} : a \in Z_p^*, b \in Z_p \} \]
  is *Universal*

• This implies that for *any* set of \( n \) keys \( K = \{ k_1, k_2, \ldots, k_n \} \), an effective way of storing the keys is to
  – Choose a random pair \((a, b)\) uniformly at random from the \( p(p-1) \) pairs in \( Z_p^* \times Z_p \)
  – Hash the items in \( K \) using hash function \( h_{a,b} \)

• Because \( H \) is Universal, average time for storing the data will be \( O(n\alpha) \) where \( \alpha = n/m \) is the load factor

• Average time for doing a search will be \( (1 + \alpha) \)
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Odds & Ends

- Hashing first recognized as a technique in the 1950’s
- Comes from English word implying *chop and mix*
- Many different types of hashing for dictionary storage out there. This introduction only scratched the surface

A *Cryptographic Hash Function* is a hash function that is *almost* impossible to invert efficiently, i.e., given $h(x)$ very difficult to find $x$.
- Almost by necessity requires that function $h$ distributes keys pretty “randomly” over $0, 1, 2, \ldots, m$. If not true, then would have first step towards guessing value of $x$ that produces $h(x)$.

- Example: Password protection. System password file only stores $h($password$)$ and not the password itself.
  * When user logs in and types password $p$, system checks $h(p)$ against file.
  * If an attacker steals the file it wouldn’t be helpful, since attacker can’t invert hashed password to get original one