B-Trees

Version of October 2, 2014
An AVL tree can be an excellent data structure for implementing dictionary search, insertion and deletion.

- Each operation on an \( n \)-node AVL tree takes \( O(\log n) \) time.
Motivation

- An AVL tree can be an excellent data structure for implementing dictionary search, insertion and deletion.
  - Each operation on an \( n \)-node AVL tree takes \( O(\log n) \) time.
  - This only works, though, long as the entire data structure fits into main memory.
An AVL tree can be an excellent data structure for implementing dictionary search, insertion and deletion.

- Each operation on an $n$-node AVL tree takes $O(\log n)$ time.

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When the data size is too large and data must reside on disk, AVL performance may deteriorate rapidly.
For a typical machine

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Consider a database with $10^9$ items (stored on disk)
- Tree would have height $\sim \log_2 10^9 = 30$
- Operations on these BSTS would need 30 disk accesses
  a very slow 0.3 second!
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- Want a way to substantially reduce number of disk accesses.
From Binary to $M$-ary

- Idea: allow the nodes to have many children
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  - More branching $\Rightarrow$ Shallower Tree $\Rightarrow$ Fewer disk accesses
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  - Each internal node has at most $m - 1$ keys
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  - Example: if $m = 100$, then $\log_{100} 10^9 < 5$
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  - Example: if $m = 100$, then $\log_{100} 10^9 < 5$
  - This reduces disk accesses and speeds up search significantly
A **B-tree of (minimum) degree** $t \geq 2$ has following properties:
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1. Every node $x$ (except root) has between $t$ and $2t$ children
   - Node with $n[x]$ keys has $n[x] + 1$ children.
     $\Rightarrow$ between $t - 1$ and $2t - 1$ keys
   - Root has at most $2t$ children
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A **B-tree of (minimum) degree** $t \geq 2$ has following properties:

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   c. \( n[x] + 1 \) pointers \( c_1[x], c_2[x], \ldots, c_{n[x]+1}[x] \) to its children
      (Leaf nodes have no children, so their \( c_i \) fields are undefined)
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4. Keys $\text{key}_i[x]$ separate ranges of keys in subtrees:
   if $k_i$ is a key stored in the subtree with root $c_i[x]$, then
   
   $$k_1 \leq \text{key}_1[x] \leq k_2 \leq \text{key}_2[x] \leq \cdots \leq \text{key}_{n[x]}[x] \leq k_{n[x]+1}$$
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\]
B-Tree Example

- $t = 2$: the simplest B-tree
- \( t = 2 \): the simplest B-tree
  - Every node has \textit{at least one key}.
    - Every internal node has \textit{at least 2 children}
B-Tree Example

- $t = 2$: the simplest B-tree
  - Every node has at least one key.
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  - Every node has at most 3 keys.
    - Every internal node has at most 4 children
$t = 2$: the simplest B-tree

- Every node has at least one key.
  - Every internal node has at least 2 children
- Every node has at most 3 keys.
  - Every internal node has at most 4 children

A node is **full** if it contains exactly $2t - 1$ keys
(e.g., nodes colored in the above example)

We choose $t$ such that an internal node fits in one disk block.
Consider the worst case

- the root contains one key
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which implies,

- 1 node at depth 0; 2 nodes at depth 1; $2t$ nodes at depth 2; $2t^2$ nodes at depth 3; ...; $2t^{h-1}$ nodes at depth $h$
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Thus, for any \( n \)-key B-tree of minimum degree \( t \geq 2 \) and height \( h \)

\[
n \geq 1 + (t - 1) \sum_{i=1}^{h} 2t^{i-1} = 1 + 2(t - 1) \left( \frac{t^h - 1}{t - 1} \right) = 2t^h - 1.
\]
Height of B-Tree

Consider the worst case

- the root contains one key
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\begin{align*}
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Therefore, \( h \leq \log_t \frac{n+1}{2} \).
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Therefore, \( h \leq \log_t \frac{n+1}{2} \).

- Compared with AVL trees, a factor of about \( \log_2 t \) is saved in the number of nodes examined for most tree operations.
• Basically follows insertion strategy of binary search tree
Basically follows insertion strategy of binary search tree
- Insert the new key into an existing leaf node
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<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
<th>E</th>
<th>F</th>
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Insert V

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Don’t. Split full nodes BEFORE inserting into them!
Insertion: How to insert into a full node?

- Don’t. **Split** full nodes BEFORE inserting into them!
- Given a **nonfull** internal node $x$, an index $i$, and a node $y$ such that $y = c_i[x]$ is a **full** child of $x$.

$$x \rightarrow_{\text{key } i-1[x]} y = c_i[x]$$

$$\begin{array}{c}
G \\
M \\
J \\
K \\
L \\
\end{array}\quad \rightarrow\quad \begin{array}{c}
G \\
K \\
M \\
J \\
L \\
\end{array}$$

$$\begin{array}{c}
T_1 \\
T_2 \\
T_3 \\
T_4 \\
\end{array}\quad \rightarrow\quad \begin{array}{c}
T_1 \\
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**Split** the full node $y$ (with $2t-1$ keys) around its median key $\text{key } t[y]$ into two nodes with $t-1$ keys each.

**Move** $\text{key } t[y]$ up into $y$’s parent $x$ to separate the two nodes.
Don’t. **Split** full nodes BEFORE inserting into them!

Given a **nonfull** internal node \( x \), an index \( i \), and a node \( y \) such that \( y = c_i[x] \) is a **full** child of \( x \).

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How can we insure that the parent of a full node is not full?
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### Answer
While moving down the tree, **split every full node** along the path from the root to the leaf where the new key will be inserted.
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- A key can be inserted into a B-tree in a single pass down the tree from the root to a leaf.
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Answer
While moving down the tree, split every full node along the path from the root to the leaf where the new key will be inserted.

- A key can be inserted into a B-tree in a single pass down the tree from the root to a leaf.
- Splitting the root is the only way to increase the height of a B-tree.
Insertion: Example

(a) initial tree

(b) insert $H$: split the encountered full node
(c) insert $H$: split the encountered full node

(d) insert $H$: insert into an existing nonfull leaf node
Trivial case: the leaf that contains the deleted key is not small (i.e., before deletion, it contains at least $t$ keys)
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Deletion Strategy

Question

How to delete key $k$ moving down from root without “backing up”?

Remove $(x, k)$ will remove $k$ from subtree rooted at $x$.

Algorithm walks down tree towards $k$. Will (for non-root $x$) first ensure that $x$ contains at least $t$ keys. Then will either remove $k$ or recursively call Remove $(x', k')$, where $k'$ is some key (possibly not $k$) and $x'$ is the root of subtree of $x$ containing $k'$.

If $k$ is in leaf $x$, condition ensures deletion is trivial.

Two other, more complicated cases, to consider.

Case 1: $k$ is in the internal node $x$.

Case 2: $k$ is not in the internal node $x$. 

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**Answer**
$Remove(x, k)$ will remove $k$ from subtree rooted at $x$. Algorithm walks down tree towards $k$. Will (for non-root $x$) first ensure that $x$ contains at least $t$ keys. Then will either remove $k$ or recursively call $Remove(x', k')$, where $k'$ is some key (possibly not $k$) and $x'$ is the root of subtree of $x$ containing $k'$.
**Deletion Strategy**

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- If $k$ is in leaf $x$, condition ensures deletion is trivial
- Two other, more complicated cases, to consider

Case 1 $k$ is in the internal node $x$
Case 2 $k$ is not in the internal node $x$
When viewing the following, note that height of tree remains unchanged except in special cases of 1(c) and 2(c) when

- root $x$ contains exactly one key
- root $x$’s two children $c_1[x]$ and $c_2[x]$ each contain exactly $t - 1$ keys.

Both 1(c) and 2(c) will then merge $c_1[x]$ key[x] and $c_2[x]$ into one new root node and then proceed to delete $k$ from the new tree rooted at this new node.

We will not explicitly illustrate these cases in the following slides.
Deletion: Case 1a

Case 1: key $k$ is in the internal node $x$

a. If the child $y$ that precedes $k$ in node $x$ has at least $t$ keys
Deletion: Case 1a

Case 1: key $k$ is in the internal node $x$

a. If the child $y$ that precedes $k$ in node $x$ has at least $t$ keys

1. find the predecessor $k'$ of $k$ in the subtree rooted at $y$
Deletion: Case 1a

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- the predecessor $F$ of $G$ is moved up to take $G$’s position
Case 1: key \( k \) is in the internal node \( x \)

b. If the child \( z \) that follows \( k \) in node \( x \) has at least \( t \) keys
Deletion: Case 1b

Case 1: key \( k \) is in the internal node \( x \)

b. If the child \( z \) that follows \( k \) in node \( x \) has at least \( t \) keys
   1. find the successor \( k' \) of \( k \) in the subtree rooted at \( z \)
Deletion: Case 1b

Case 1: key $k$ is in the internal node $x$

b. If the child $z$ that follows $k$ in node $x$ has at least $t$ keys

1. find the successor $k'$ of $k$ in the subtree rooted at $z$
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b. If the child \( z \) that follows \( k \) in node \( x \) has at least \( t \) keys

1. find the successor \( k' \) of \( k \) in the subtree rooted at \( z \)
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3. recursively delete \( k' \)

the successor \( J \) of \( F \) is moved up to take \( F \)'s position
Deletion: Case 1c

Case 1: key $k$ is in the internal node $x$

   c. If both $y$ and $z$ have only $t - 1$ keys
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c. If both $y$ and $z$ have only $t - 1$ keys

1. merge $k$ and $z$ into $y$ ($y$ now contains $2t - 1$ keys)
Deletion: Case 1c

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2. recursively delete $k$ from $y$
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c. If both $y$ and $z$ have only $t - 1$ keys

1. merge $k$ and $z$ into $y$ ($y$ now contains $2t - 1$ keys)
2. recursively delete $k$ from $y$

$L$ deleted (trivial case)

$J$ is pushed down to make node $DJK$, from where $J$ is deleted
Deletion: Case 2a

Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has $> t − 1$ keys then

- Recursively delete $k$ from $c_i[x]$. 
Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has only $t - 1$ keys...
Deletion: Case 2b

Case 2: the key \( k \) is not in the internal node \( x \), then determine the root \( c_i[x] \) whose subtree contains \( k \). If \( c_i[x] \) has only \( t - 1 \) keys

b. If \( c_i[x] \) has an immediate sibling with at least \( t \) keys
Deletion: Case 2b

Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has only $t - 1$ keys

b. If $c_i[x]$ has an immediate sibling with at least $t$ keys

1. give $c_i[x]$ an extra key by moving a key from $x$ down into $c_i[x]$
Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has only $t-1$ keys

b. If $c_i[x]$ has an immediate sibling with at least $t$ keys

1. give $c_i[x]$ an extra key by moving a key from $x$ down into $c_i[x]$
2. move a key from $c_i[x]$’s immediate left or right sibling up into $x$
Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has only $t - 1$ keys

b. If $c_i[x]$ has an immediate sibling with at least $t$ keys
   1. give $c_i[x]$ an extra key by moving a key from $x$ down into $c_i[x]$
   2. move a key from $c_i[x]$’s immediate left or right sibling up into $x$
   3. move the appropriate child pointer from the sibling into $c_i[x]$
Deletion: Case 2b

Case 2: the key \( k \) is not in the internal node \( x \), then determine the root \( c_i[x] \) whose subtree contains \( k \). If \( c_i[x] \) has only \( t - 1 \) keys

b. If \( c_i[x] \) has an immediate sibling with at least \( t \) keys

1. give \( c_i[x] \) an extra key by moving a key from \( x \) down into \( c_i[x] \)
2. move a key from \( c_i[x] \)'s immediate left or right sibling up into \( x \)
3. move the appropriate child pointer from the sibling into \( c_i[x] \)
4. recursively delete \( k \) from the appropriate child of \( x \)
Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has only $t-1$ keys

b. If $c_i[x]$ has an immediate sibling with at least $t$ keys

1. give $c_i[x]$ an extra key by moving a key from $x$ down into $c_i[x]$ 
2. move a key from $c_i[x]$’s immediate left or right sibling up into $x$
3. move the appropriate child pointer from the sibling into $c_i[x]$
4. recursively delete $k$ from the appropriate child of $x$
Deletion: Case 2c

Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has only $t - 1$ keys and both of its immediate siblings have $t - 1$ keys
Deletion: Case 2c

Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has only $t-1$ keys, and both of its immediate siblings have $t-1$ keys, merge $c_i[x]$ with one sibling.
Case 2: the key $k$ is not in the internal node $x$, then determine the root $c_i[x]$ whose subtree contains $k$. If $c_i[x]$ has only $t - 1$ keys
   c. and both of its immediate siblings have $t - 1$ keys
      1. merge $c_i[x]$ with one sibling
      2. recursively delete $k$ from the appropriate child of $x$
Deletion: Case 2c

Case 2: the key \( k \) is not in the internal node \( x \), then determine the root \( c_i[x] \) whose subtree contains \( k \). If \( c_i[x] \) has only \( t - 1 \) keys

1. merge \( c_i[x] \) with one sibling
2. recursively delete \( k \) from the appropriate child of \( x \)

\[
\begin{array}{c}
P \\
\downarrow & \\
DM & TX \\
\downarrow & \downarrow \\
C & K & NO & QR & U & W & Y & Z \\
\downarrow & \\
Delete C \\
\downarrow \\
\begin{array}{c}
P \\
\downarrow & \\
M & TX \\
\downarrow & \downarrow \\
DK & NO & QR & U & W & Y & Z
\end{array}
\end{array}
\]

- \( D \) is pushed down to get node \( CDK \), from where \( C \) is deleted
Saw how to maintain a $B$-tree using $\log_t n$ “operations”
- each operation requires constant number of disk reads.
- could also require many internal memory operations

For “large” $t$; useful for storing large databases on disk
- with each node a disk page

$B$-Trees created by Bayer and McCreight at Boeing in 1972

$B^+$ tree variant keeps data keys in leaves

Simplest $B$-Tree is $(2 - 3 - 4)$-tree
- Balanced tree good for internal memory storage

Another variation is $(a, b)$-trees: all non-root nodes have between $a$ and $b$ children
- Is a $B$-tree if $b = 2a$.
- Smallest (non $B$-Tree) version is $(2, 3)$-trees