

A Generic Top-Down Dynamic-Programming Approach to Prefix-Free Coding

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Limit on # of distinct code lengths,

Limit on # of 1's used (*Sound of Silence*), etc.,

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This talk: a simple technique for speeding up
the DP for many prefix-free coding variants.

- Introduction
 - A Quick Review of Prefix-Free Coding
 - New Results
- The Basic Top-Down Dynamic Programming Technique
- The Speedup
- Conclusion & Comments

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e.g., **010** is a prefix of **01011001**

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e.g., 010 is a prefix of 01011001
- W is **prefix-free** if $\forall w, w' \in W$, w is not a prefix of w' .

E.g., $\begin{matrix} 10 \\ 11 \\ 01 \end{matrix}$ is prefix free; $\begin{matrix} 1 \\ 11 \\ 01 \end{matrix}$ is not prefix-free;

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- Same problem as finding a tree with n leaves weighted by P that minimizes **weighted external path length**.

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Correspondence between codes on $\Sigma = \{0, 1\}$ and binary trees.

(or codes on general Σ and $|\Sigma|$ -ary trees)

Let 0 denote a left edge and 1 a right edge.

Codewords are leaves; Create paths to all codewords.

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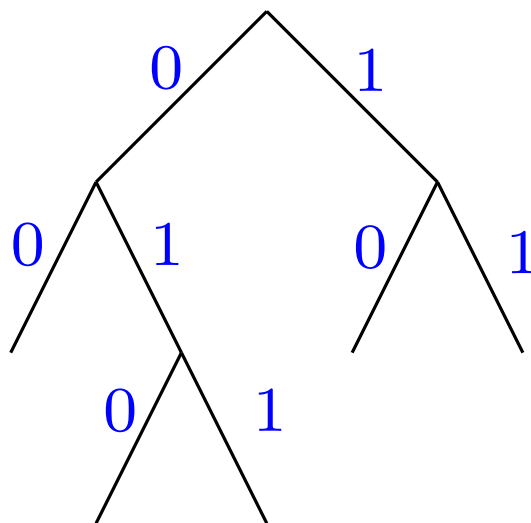
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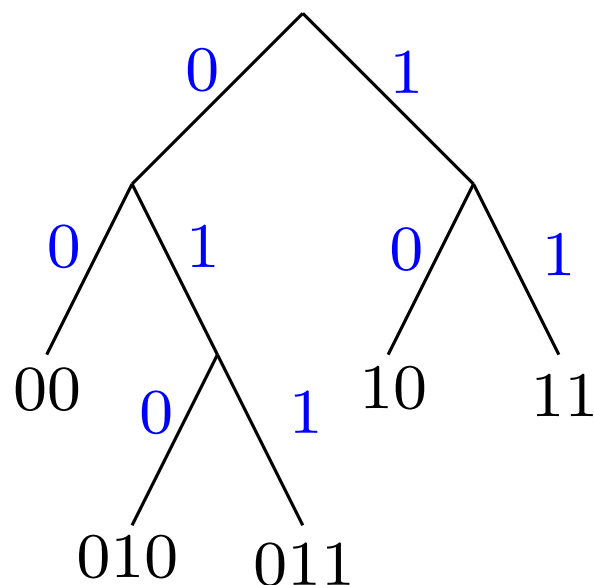
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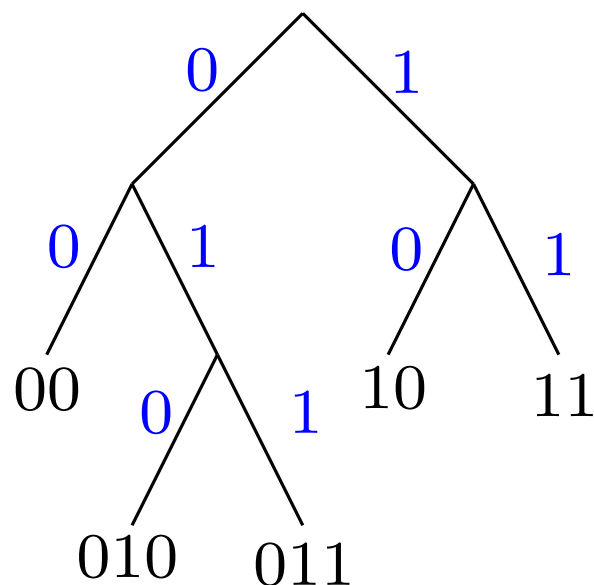
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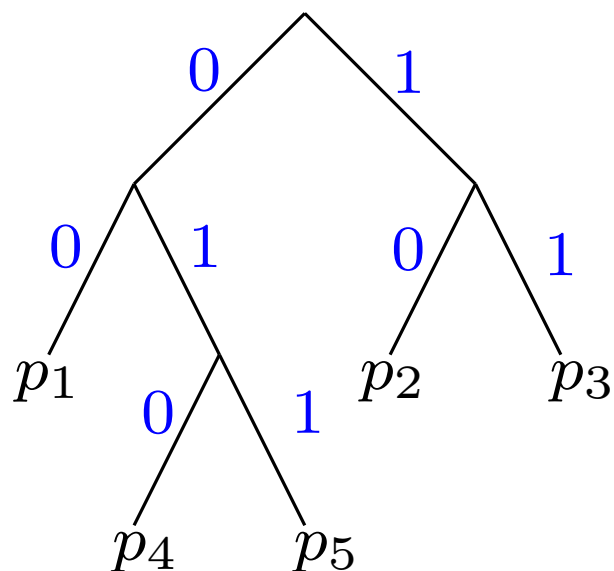
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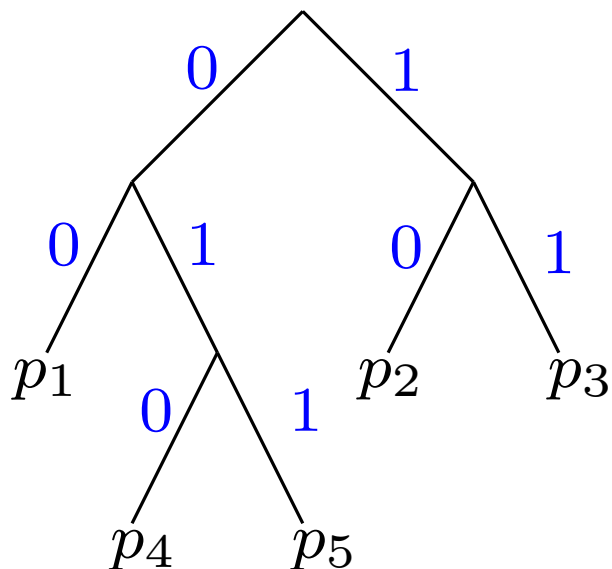
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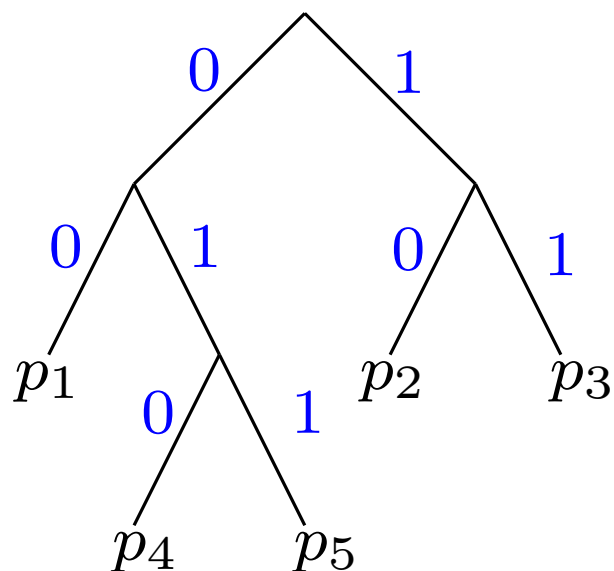
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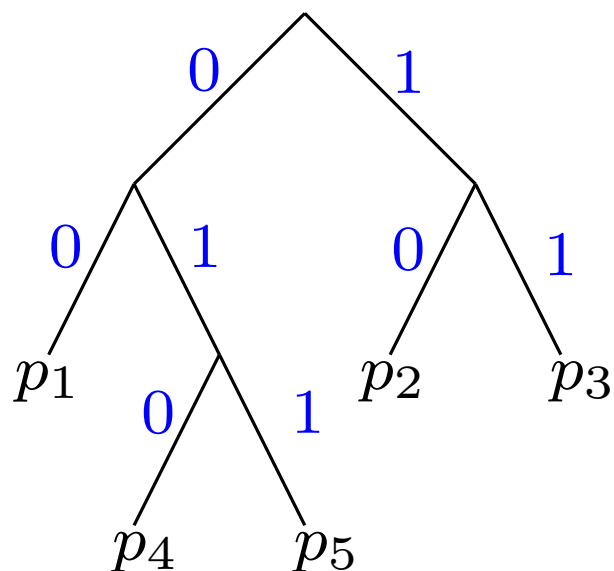
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Change problem to

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- Mixed-Radix Coding:
Size of alphabet depends upon position of character within codeword,
e.g., arity of node depends upon level in the tree.

New Results

With exception of Length-Limited Coding (which takes advantage of Schieber's (1998) min-cost length-limited paths in Monge-graphs result) we improve the DP-based algorithms for all problems on previous page.

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[1]: *Chu and Gill (1992)*

[2]: *Baer (2008)*

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This leads to lower amortized time per optimal-cost calculation.

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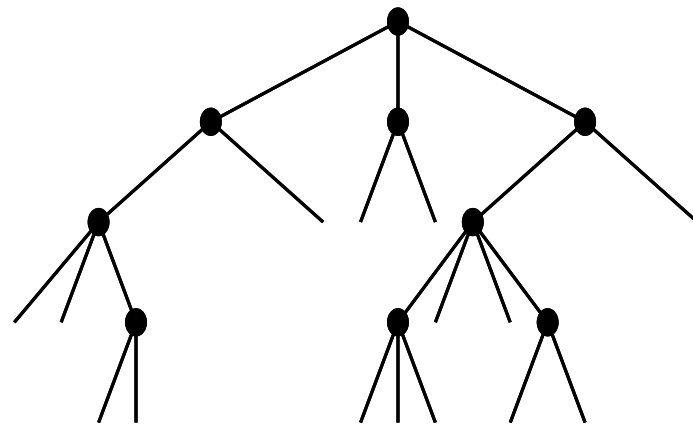
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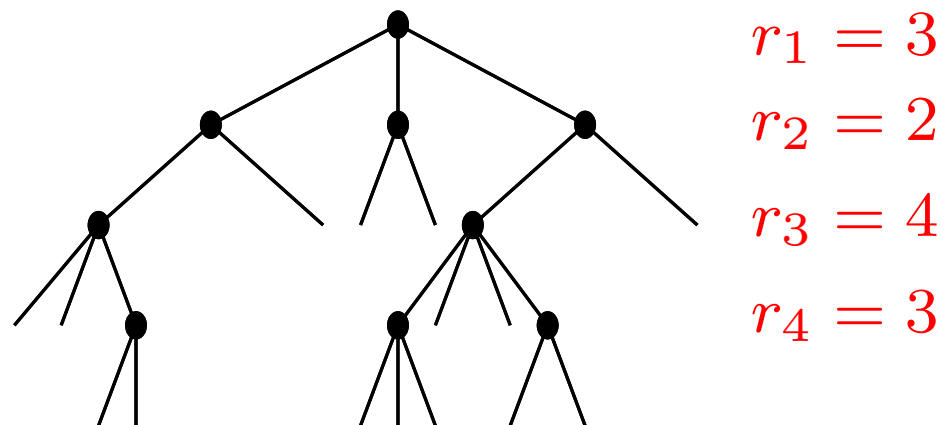


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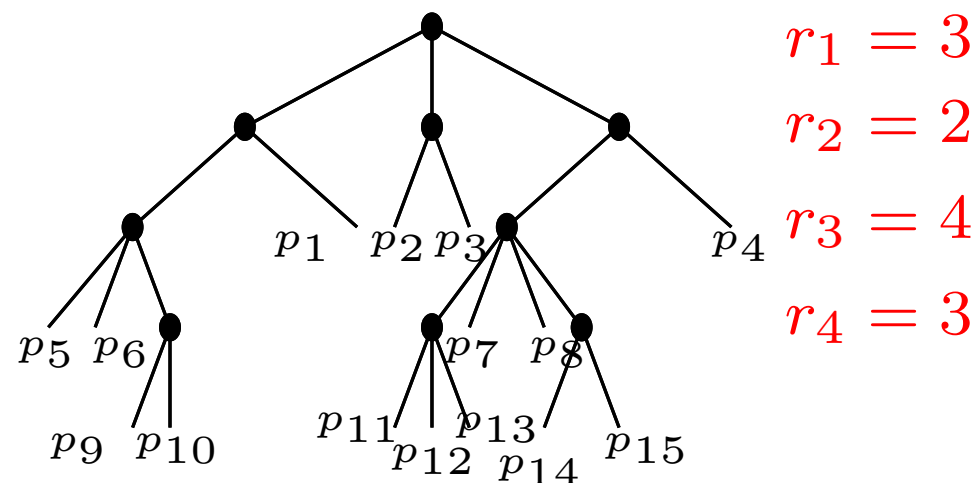
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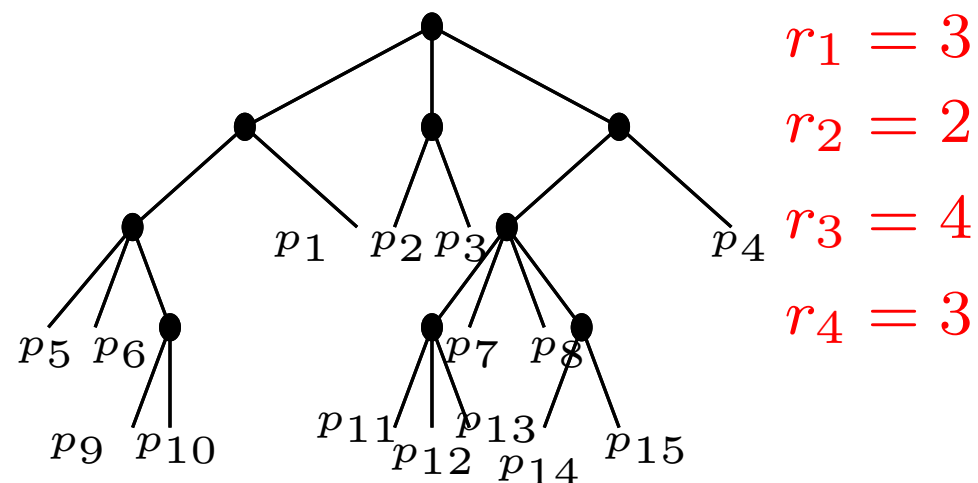
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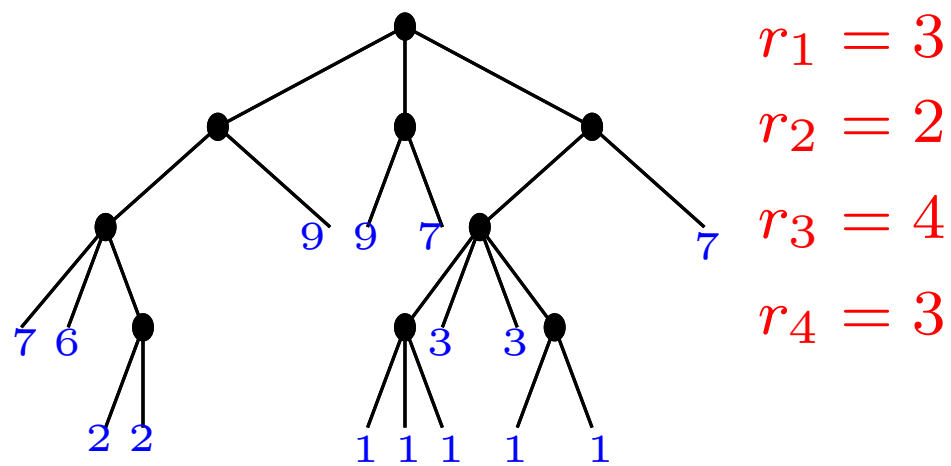
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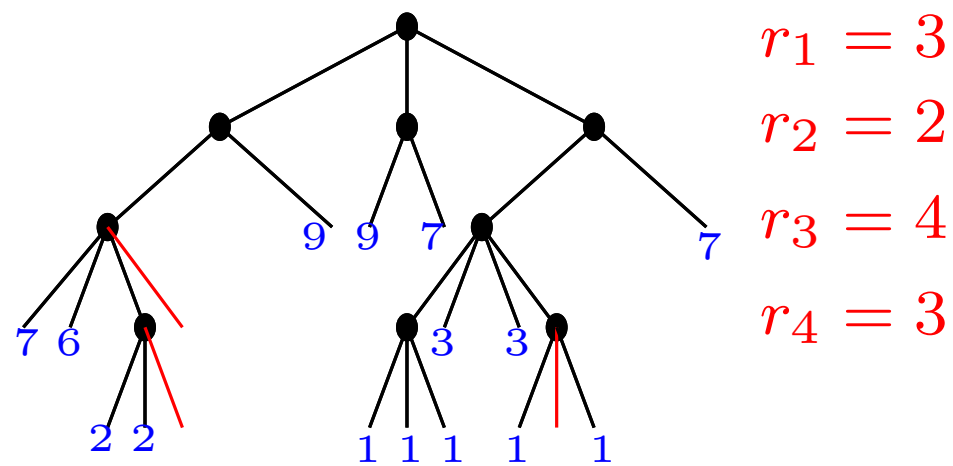
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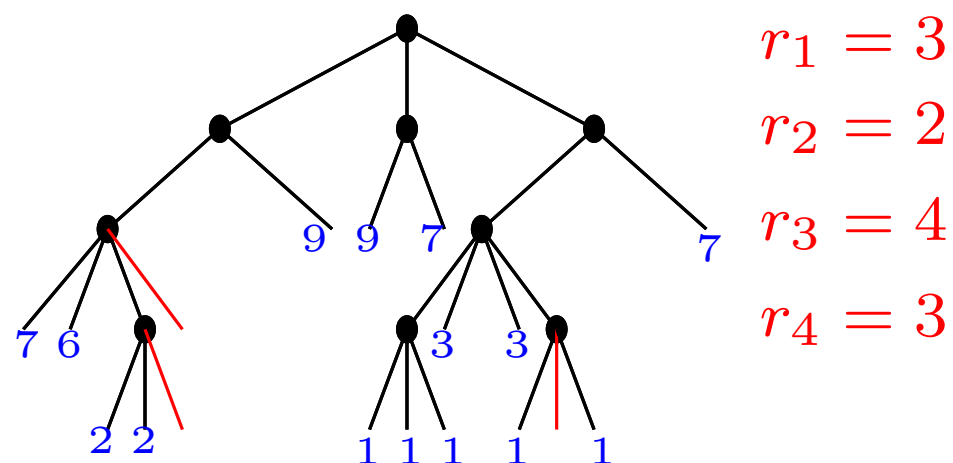
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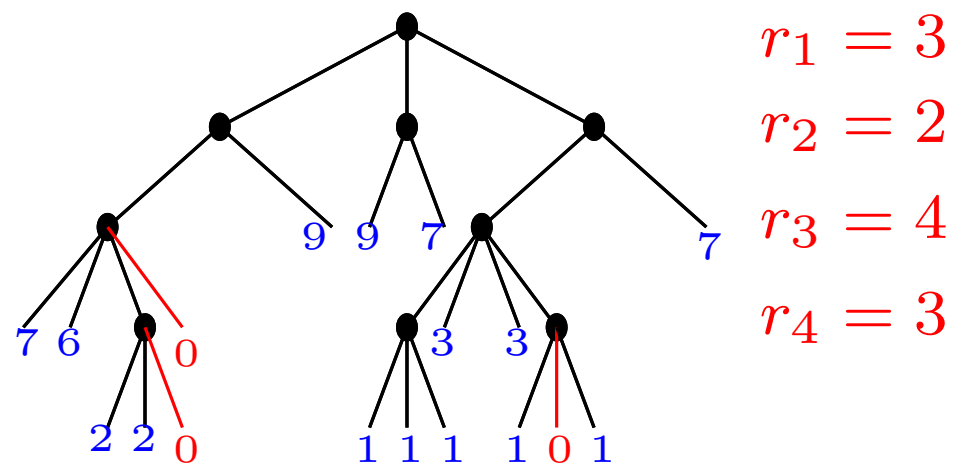
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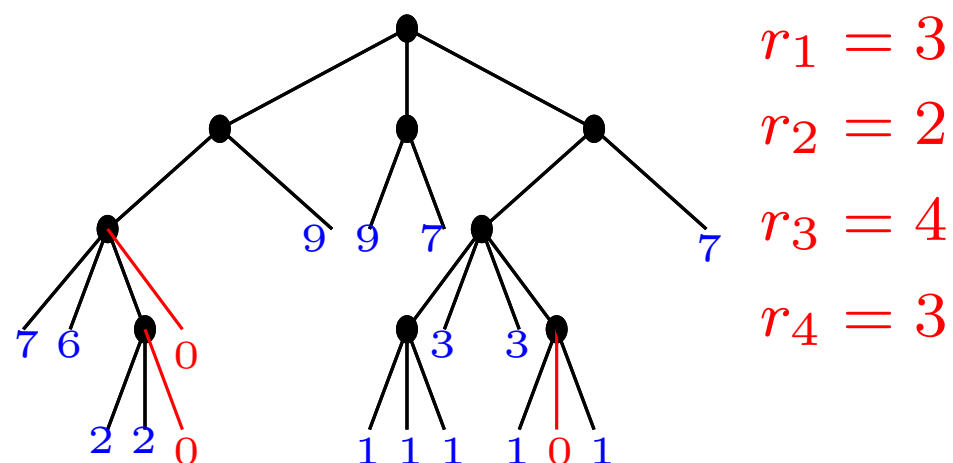
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W.L.O.G. assume that the p_i are sorted in non-increasing order

W.L.O.G. also assume that all internal nodes have exactly r_i children.

Can ensure this by padding P with arbitrarily many 0s.

(might require moving some p_i 's up the tree)

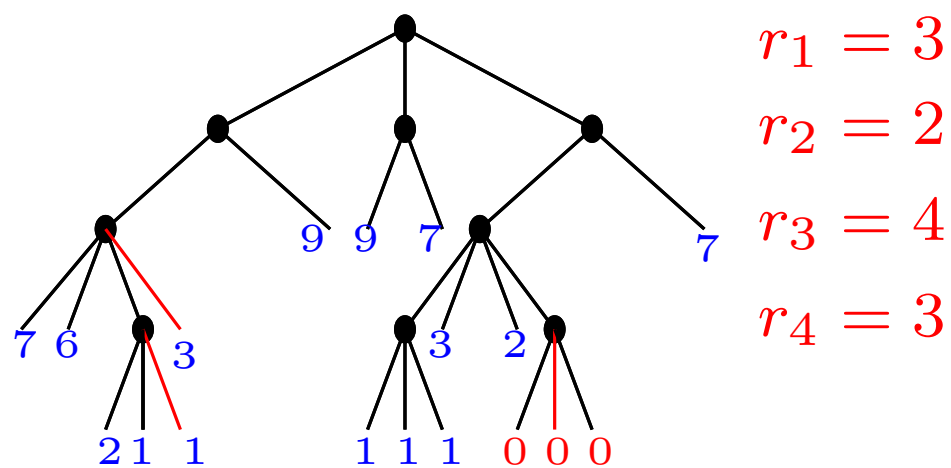
The Technique

We illustrate the technique by showing how to speed up *mixed-radix* coding from $O(n^4)$ down to $O(n^3)$. The same technique, with various bells and whistles added, speeds up all of the other problems.

In mixed-radix coding, input is weight set $P = \{p_1, \dots, p_n\}$ and *arity* list $R = \{r_1, r_2, r_3, \dots\}$.

Nodes on level $i - 1$, have arity $\leq r_i$.

Want to find tree satisfying R with minimum cost $\sum_{i=1}^n p_i d(v_i)$.



W.L.O.G. assume that the p_i are sorted in non-increasing order

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The Technique

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Build the tree *top-down*, level-by-level, using DP.

Standard technique: e.g., Golin & Rote '98, Dolev, Korach & Yukelson '99, Chan & Golin '00, Baer '08

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Idea is to keep track, at depth i , of

m : # of leaves so far

b : # of “internal” depth i nodes.

These are nodes that will be
“expanded” at next step

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m : # of leaves so far



d	m	b
0	0	1

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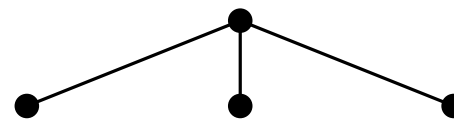
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d	m	b
0	0	1
1	0	3

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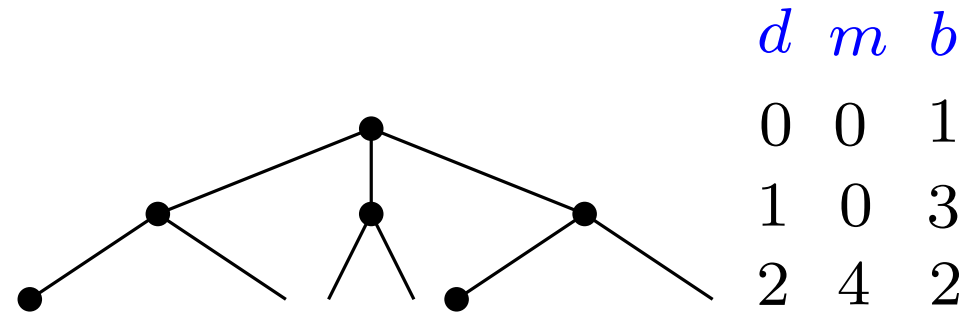
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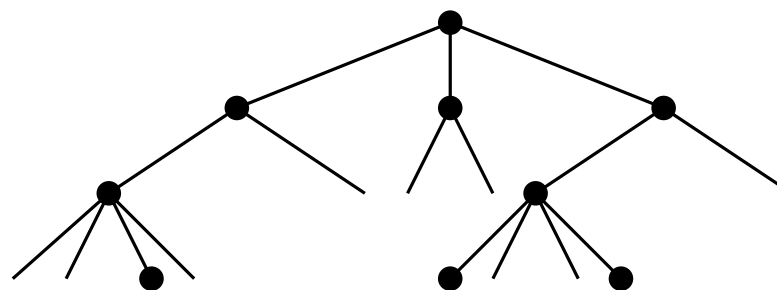
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d	m	b
0	0	1
1	0	3
2	4	2
3	9	3

The Technique

Build the tree *top-down*, level-by-level, using DP.

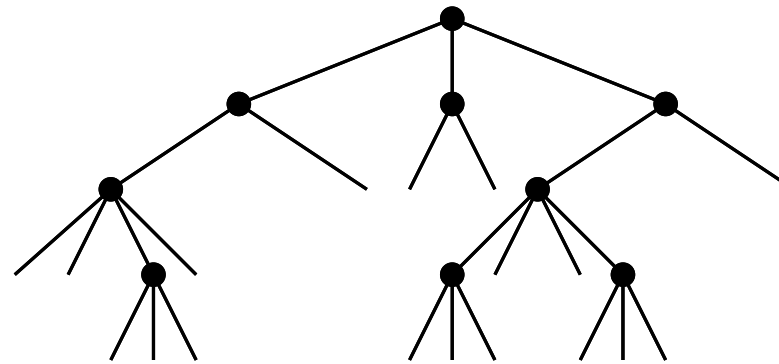
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d	m	b
0	0	1
1	0	3
2	4	2
3	9	3
4	18	0

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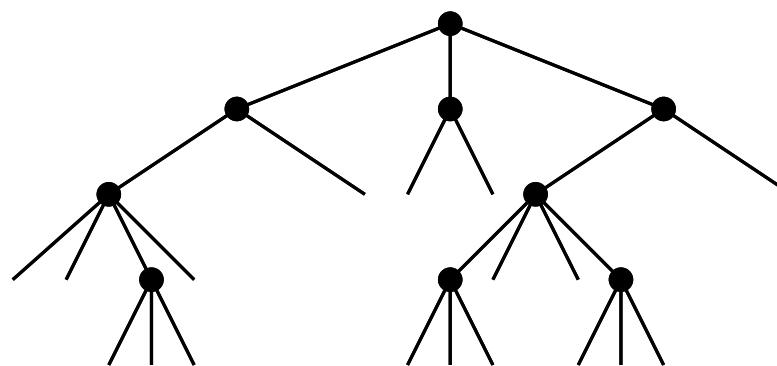
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d	m	b
0	0	1
1	0	3
2	4	2
3	9	3
4	18	0

Will also keep track of cost “so far” .

$$\sum_{t=1}^m p_i d_i + i \sum_{t>m} p_t$$

The Technique

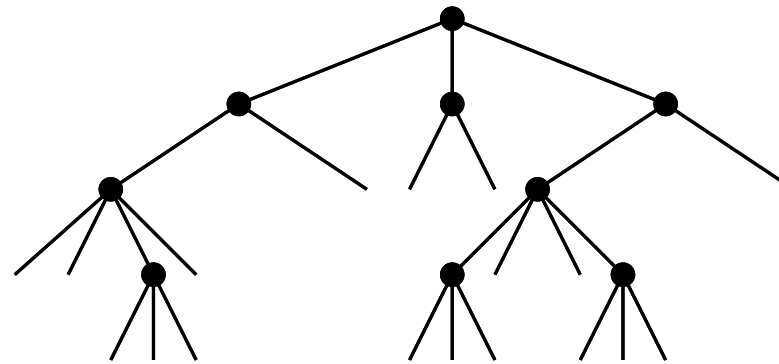
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m : # of leaves so far

d	m	b
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1	0	3
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$$\sum_{t=1}^m p_i d_i + i \sum_{t>m} p_t$$

Ex: $P = \{3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, \dots\}$

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Idea is to keep track, at depth i , of

m : # of leaves so far



d	m	b	c
0	0	1	0
1	0	3	
2	4	2	
3	9	3	
4	18	0	

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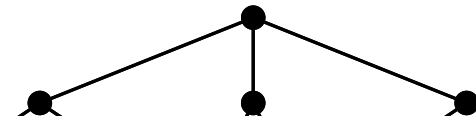
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Idea is to keep track, at depth i , of

m : # of leaves so far



d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	
3	9	3	
4	18	0	

b : # of “internal” depth i nodes.

These are nodes that will be

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Will also keep track of cost “so far” .

$$\sum_{t=1}^m p_t d_t + i \sum_{t>m} p_t$$

Ex: $P = \{ \underline{3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, \dots} \}$

$X1$

The Technique

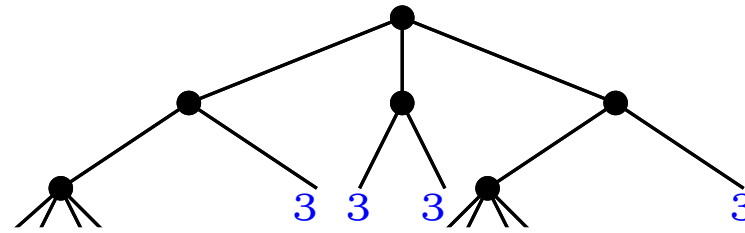
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d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	60
3	9	3	
4	18	0	

Will also keep track of cost “so far” .

$$\sum_{t=1}^m p_i d_i + i \sum_{t>m} p_t$$

Ex: $P = \{3, 3, 3, 3, 3, \underbrace{2, 2, 2, 2, 2}_{\times 2}, 1, 1, 1, 1, 1, 0, 0, \dots\}$

The Technique

Build the tree *top-down*, level-by-level, using DP.

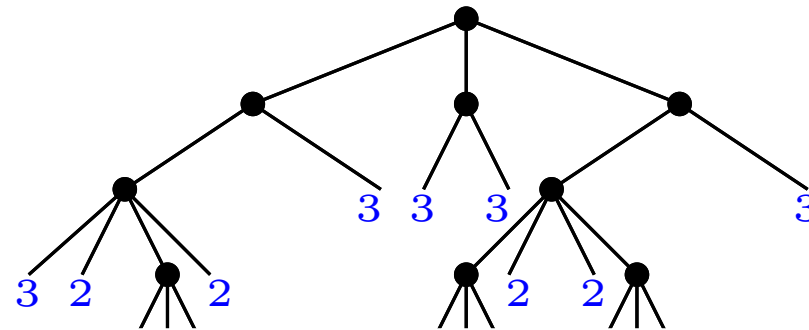
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These are nodes that will be “expanded” at next step



d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	60
3	9	3	78
4	18	0	

Will also keep track of cost “so far” .

$$\sum_{t=1}^m p_i d_i + i \sum_{t>m} p_t$$

Ex: $P = \{3, 3, 3, 3, 3, 2, 2, 2, 2, 2, \underbrace{1, 1, 1, 1, 1}_{\times 3}, 0, 0, \dots\}$

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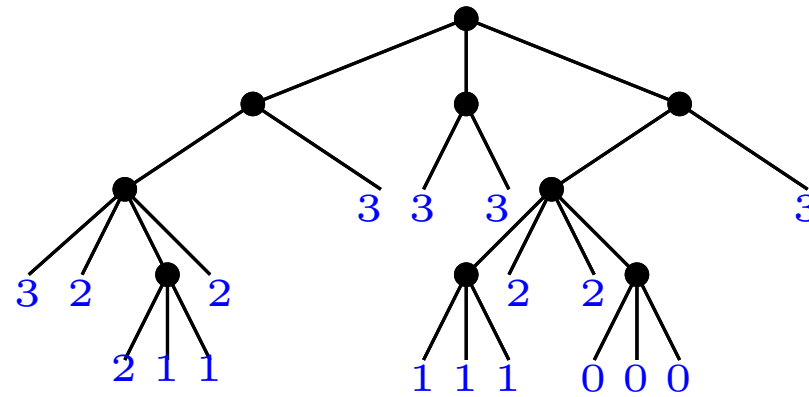
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Idea is to keep track, at depth i , of

m : # of leaves so far

b : # of “internal” depth i nodes.

These are nodes that will be “expanded” at next step



d	m	b	c
0	0	1	0
1	0	3	30
2	4	2	60
3	9	3	78
4	18	0	84

Will also keep track of cost “so far” .

$$\sum_{t=1}^m p_i d_i + i \sum_{t>m} p_t$$

Ex: $P = \{3, 3, 3, 3, 3, 2, 2, 2, 2, 2, 1, 1, 1, 1, 1, 0, 0, \dots\}$

The Technique

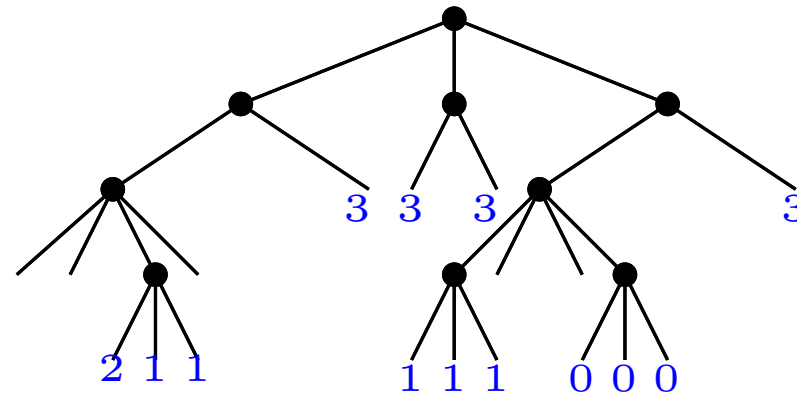
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0	0	1	0
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4	18	0	84

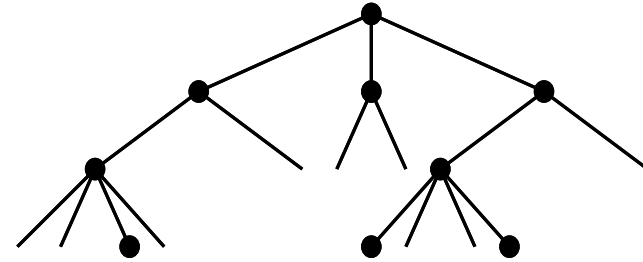
Will also keep track of cost “so far”.

$$\sum_{t=1}^m p_i d_i + i \sum_{t>m} p_t$$

If $m \geq n$, then “cost so far” is real cost of tree

The Technique

T is an i -level tree if $d(T) \leq i$.



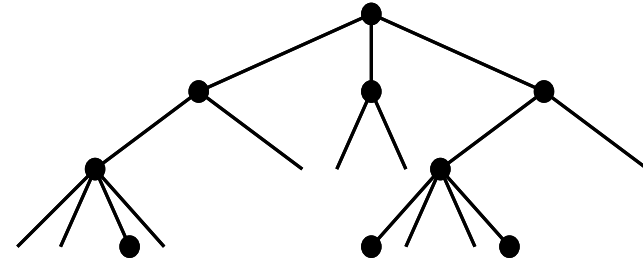
The Technique

T is an i -level tree if $d(T) \leq i$.

$$\text{sig}_i(T) = (m, b)$$

$m = \#$ leaves at depth $\leq i$.

$b = \#$ internals at depth i .



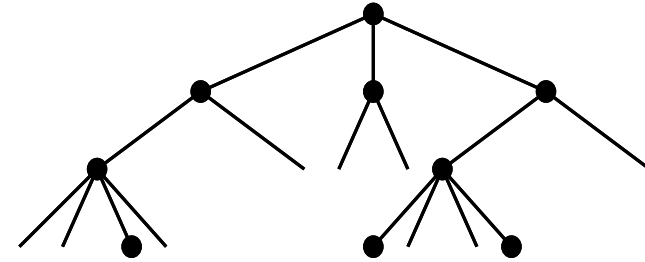
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$$\text{sig}_2(T) = (9, 3)$$

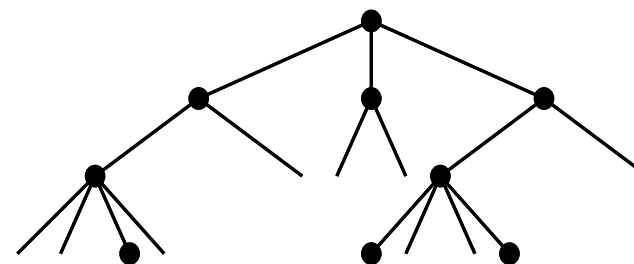
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$$\text{sig}_2(T) = (9, 3)$$

$$OPT^i[m, b] = \min [\text{cost}_i(T) \mid \text{sig}_i(T) = (m, b)].$$

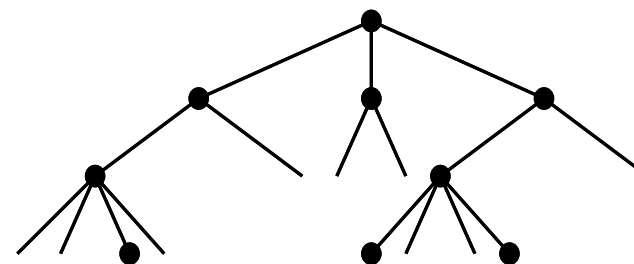
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$$OPT^i[m, b] = \min [\text{cost}_i(T) \mid \text{sig}_i(T) = (m, b)].$$

$$\min_{m \geq n} (OPT^i(m, 0))$$

is cost of min-cost tree with at least n leaves and depth $\leq i$.

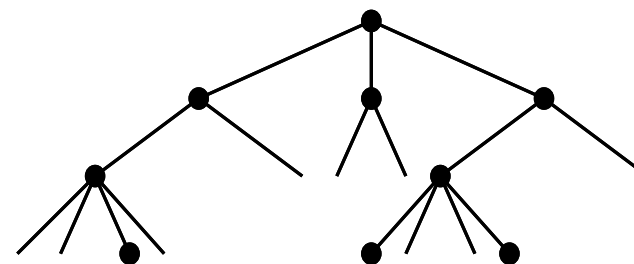
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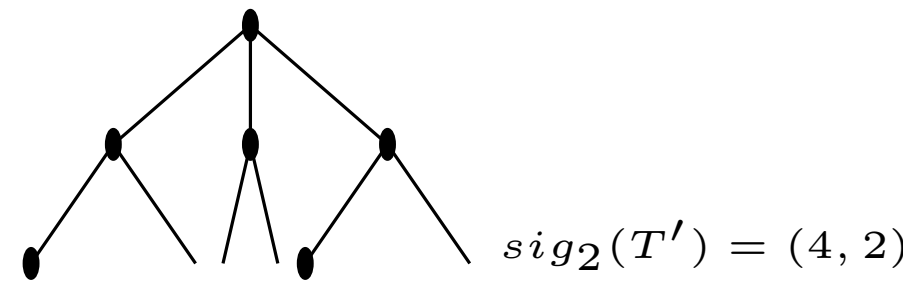
$$OPT^i[m, b] = \min [cost_i(T) \mid \text{sig}_i(T) = (m, b)].$$

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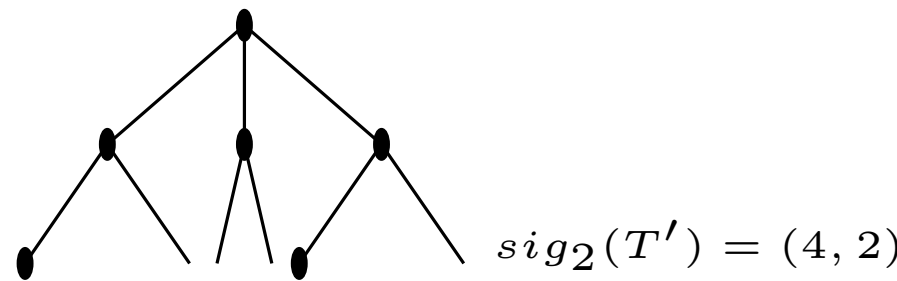
Goal: Find $\min_{m \geq n} (OPT^n(m, 0))$ and tree that achieves it

Let T' be an $(i - 1)$ -level tree with $sig_{i-1}(T) = (m', b')$.



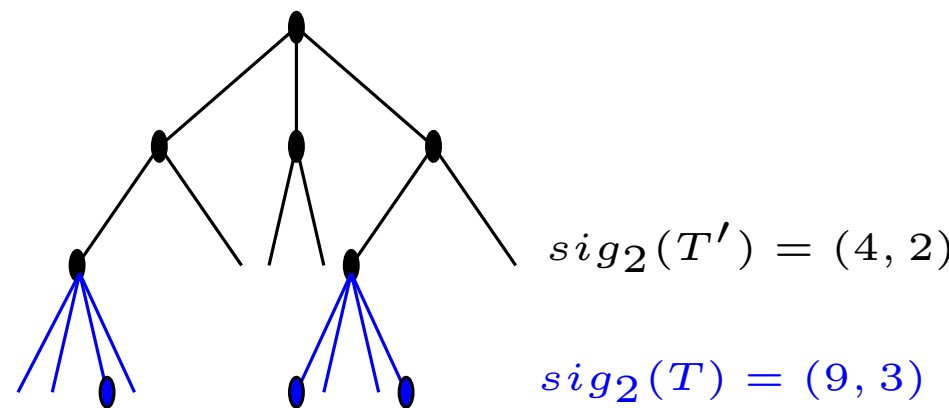
Let T' be an $(i - 1)$ -level tree with $sig_{i-1}(T) = (m', b')$.

T' is expanded to an i level tree T by adding the $r_i b'$ children on level i and choosing b of them to be internal.



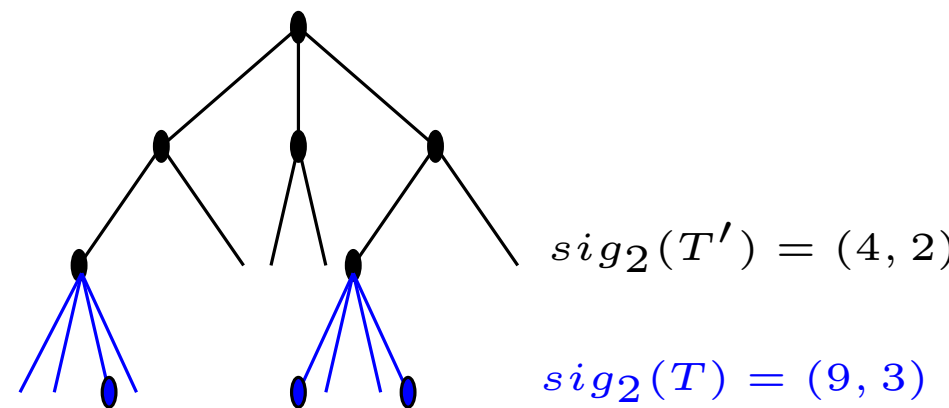
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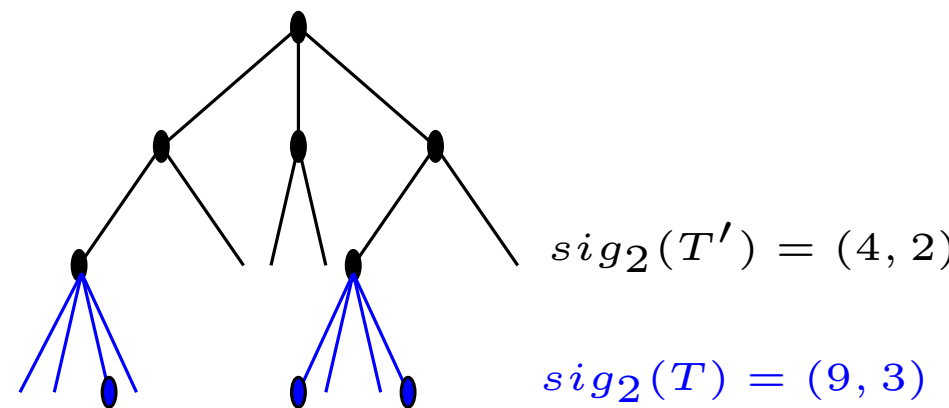
T' is expanded to an i level tree T by adding the $r_i b'$ children on level i and choosing b of them to be internal.



Lemma: $m = m' + b' r_i - b$ and $cost_i(T) = cost_{i-1}(T') + \sum_{t > m'} p_t$.

Let T' be an $(i - 1)$ -level tree with $sig_{i-1}(T') = (m', b')$.

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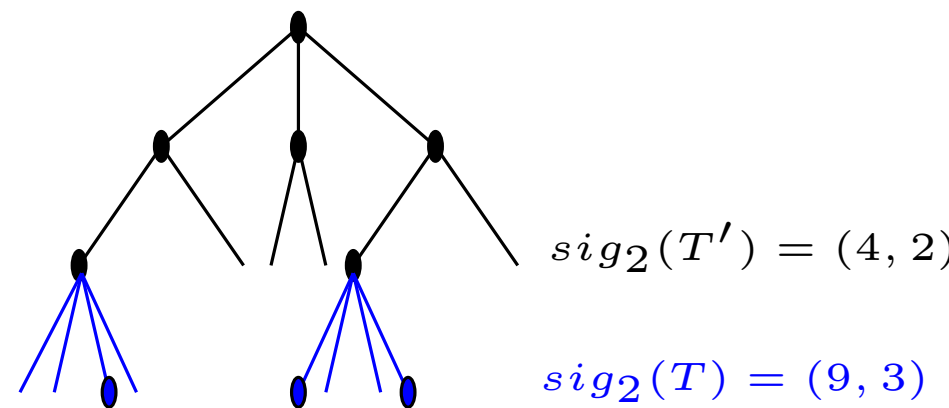


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We say that $(m', b') \xrightarrow{i} (m, b)$ if $\exists T', T$ as above.

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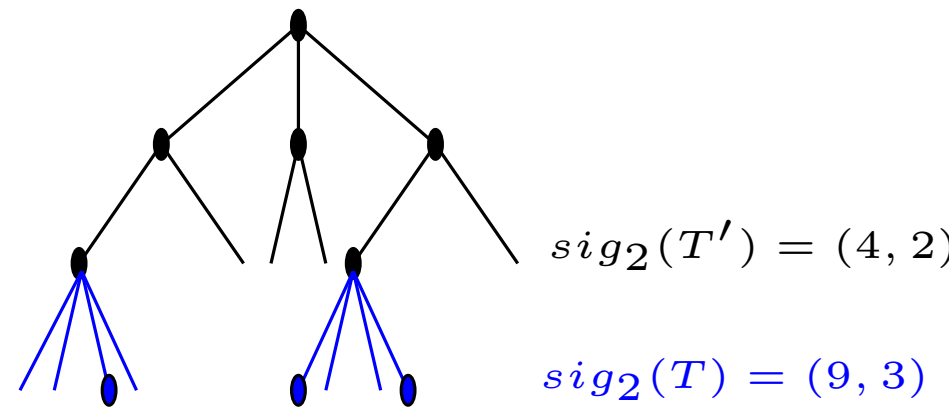


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The DP recurrence is thus

$$OPT^i[m, b] = \min_{\{(m', b') \mid (m', b') \xrightarrow{i} (m, b)\}} \{OPT^{i-1}[m', b'] + W_{m'}\}.$$

with initial condition $OPT^0[0, 1] = 0$.

- Introduction
 - A Quick Review of Prefix-Free Coding
 - New Results
- The Basic Top-Down Dynamic Programming Technique
- The Speedup
- Conclusion & Comments

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So, only need to check $O(m)$ entries to calculate given $OPT^i[m, b]$.

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So, only need to check $O(m)$ entries to calculate given $OPT^i[m, b]$.

Not hard to prove that

if $b > 0$ then $m + b \leq n$ and

if $b = 0$ then $m < n + r_i$.

So, only need to fill in $O(n^2)$ entries.

Note: paper shows how to make $O(n^2)$ independent of r_i

\Rightarrow Total time to fill in $OPT^i[,]$ table is $O(n^3)$.

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Filling in *all* of the tables and solving the entire problem in $O(n^3)$ time.

$$OPT^i[m, b] = \min_{\{(m', b') \mid (m', b') \xrightarrow{i} (m, b)\}} \{OPT^{i-1}[m', b'] + W_{m'}\}.$$

where $m = m' + b'r_i - b$ and $b \leq b'r_i$.

$$OPT^i[m, b] = \min_{\{(m', b') \mid (m', b') \xrightarrow{i} (m, b)\}} \left\{ \cancel{OPT^{i-1}[m', b'] + W_{m'}} \right\}.$$

$X[m', b']$

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where $m = m' + b'r_i - b$ and $b \leq b'r_i$.

For fixed $d \geq 1$ set

$$\mathcal{I}(d) = \{(m, b) \mid m + b = d\}, \quad \mathcal{I}'_i(d) = \{(m', b') \mid m' + b'r_i = d\}.$$

$$OPT^i[m, b] = \min_{\{(m', b') \mid (m', b') \xrightarrow{i} (m, b)\}} \left\{ \cancel{OPT^{i-1}[m', b'] + W_{m'}} \right\}.$$

where $m = m' + b'r_i - b$ and $b \leq b'r_i$.

For fixed $d \geq 1$ set

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To fully solve the problem, we must fill in,

$$OPT^1[m, b], OPT^2[m, b], \dots, OPT^n[m, b].$$

From above this takes only $O(n^3)$ time,

improving upon the old bound of $O(n^4 \log n)$.

- Introduction
 - A Quick Review of Prefix-Free Coding
 - New Results
- The Basic Top-Down Dynamic Programming Technique
- The Speedup
- Conclusion & Comments

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Reserved Length Coding (i)	$O(gn^3)$ [2]	$O(gn^2)$
Reserved Length Coding (ii)	$O(g^3 n^3 \log^g n)$ [2]	$O(gn^2 \log n)$
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In reality, all of the above problems can be solved using a top-down DP very similar to the one for mixed-radix coding. The major problem-specific change is in the definition of signature.

Furthermore, almost the same type of batching technique, e.g., defining similar $\mathcal{I}(d)$ and $\mathcal{I}'(d)$ and showing that $OPT[m, b]$ for $(m, b) \in \mathcal{I}(d)$ only depend upon values in $\mathcal{I}'(d)$, holds for all of these problems.

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The batching technique was later shown to be applicable to other coding problems, such as 1-ended coding, that do not (at least obviously) possess the Monge property.

What other problems can this type of batching speed up?