Lecture 2: Sets, Relations, and Functions (a Review)

1. Sets

- A *set* is a collection of objects.
- Let $L = \{a, b, c, d\}$ be a set of four *elements*.

  We use $a \in L$ to denote that a is *in* L; and $z \notin L$ to denote that z is not in L.
- The *empty set*, denoted by $\emptyset$, contains zero elements. All other sets are called *non-empty*.
- A set is *finite* if it has a finite number of elements. Otherwise, the set is *infinite*.
- A is a *subset* of B, $A \subseteq B$
  if every element in A is an element in B.
- Two sets A and B are *equal* ($A = B$)
  if $A \subseteq B$ and $B \subseteq A$.
- If $A \subset B$ (and $A \neq B$), then A is a *proper subset* of B. (By this definition $\emptyset$ is a proper subset of any non-empty set.)

Set operations. Let A and B be sets.

- *Intersection*,

  $$A \cap B = \{x : x \in A \text{ and } x \in B\}.$$

  If $A \cap B = \emptyset$, A and B are *disjoint*.

- *Union*, $A \cup B = \{x : x \in A \text{ or } x \in B\}$.
- *Difference*, $A - B = \{x : x \in A \text{ and } x \notin B\}$.

Power sets and partition.

- The set of all subsets of a set A, denoted by 2^A, is called the *power set* of A. For example,

  $$2^{\{c,d\}} = \{\emptyset, \{c\}, \{d\}, \{c, d\}\}.$$

- A *partition* of A is any set of non-empty subsets of A, denoted by $A_1, A_2, \cdots$, such that
  1. $A_1 \cup A_2 \cup \cdots = A$
  2. and for all $i \neq j$, $A_i \cap A_j = \emptyset$ (i.e. mutually disjoint).

2. Relations and Functions

Cartesian product of A and B, denoted by $A \times B$, is the set of all possible *ordered pairs* (a, b) with $a \in A$ and $b \in B$.

$$\{1, 2\} \times \{3, 4\} = \{(1, 3), (1, 4), (2, 3), (2, 4)\}.$$

A **binary relation** on A and B is a subset of $A \times B$.

A **function** from A to B, or a binary function, $f : A \rightarrow B$, is a binary relation R on A and B such that

$$\forall a \in A, \quad \text{if}(a, b) \in R \text{ and } (a, c) \in R, \text{ then } b = c.$$

(A is called the *domain* of f and B is called its *range*.)

Let $f = R = \{(1, 3), (2, 4)\} \subseteq A \times B$. Then, $f(1) = 3$ and $f(2) = 4$.

A n-ary relation and function can be defined similarly.

Bijection.

- $f : A \to B$ is *one-one* if for
$$\forall a \neq a' \in A, \qquad f(a) \neq f(a').$$
- $f : A \to B$ is *onto* if for all $b \in B$, there exists some $a \in A$ such that $f(a) = b$.
- f is a *bijection* if it is both one-one and onto.
- Example:

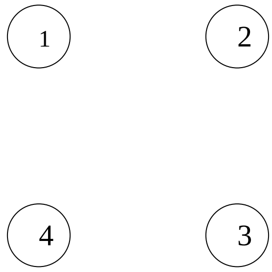
  Let $A = \{1, 2\}$ and $B = \{3, 4\}$.
  $f_1 = \{(1, 3), (2, 4)\}$ is a bijection,
  while $f_2 = \{(1, 4), (2, 4)\}$ is neither one-one nor onto.

Reflexivity, symmetry, and transitivity. Let $R \subseteq A \times A$ be a relation.

- R is *reflexive* if $(a, a) \in R$ for each $a \in R$.
- R is *symmetric* if $(b, a) \in R$ whenever $(a, b) \in R$.
- R is *transitive* if $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$.
- R is an *equivalence relation* if it is reflexive, symmetric and transitive.

Let $A = \{1, 2, 3, 4\}$. $R \subseteq A \times A$ can be represented by a directed graph where an arrow is drawn from a to b if and only if $(a, b) \in R$.

Draw the directed graph to represent the equivalence relation $\{(1, 1), (2, 2), (3, 3), (4, 4), (1, 2), (2, 1), (1, 3), (3, 1), (2, 3), (3, 2)\}$:



Closure. Let D be any set, $D^n = \overbrace{D \times D \cdots \times D}^{n}$, and let $f : D^n \to D$.

- A subset $B \subset D$ is *closed under* f if $f(b_1, \cdots, b_n) \in B$ for *all* $b_1, \cdots, b_n \in B$.

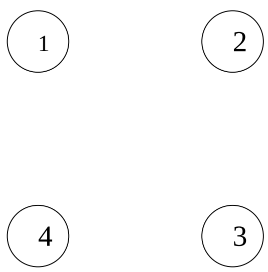
- Example

  Let $D = \mathcal{Z}$ and $B = \mathcal{N}$. B is *closed* under addition but not under subtraction.

The **reflexive, transitive closure** of a binary relation $R \subseteq A \times A$, denoted by R^*, is

$R \cup \{(a, b)$: there is a chain in R from a to b in the directed graph representing R.$\}$

For example, let $R = \{(1, 2), (2, 3), (3, 4)\}$.
Then $R^* = R \cup \{(1, 1), (2, 2), (3, 3), (4, 4), (1, 3), (1, 4), (2, 4)\}$.



Infinite sets.

- The number of elements in a set A is its *cardinality*, denoted by $|A|$.
- If $|A|$ is finite, then A is a finite set. Otherwise, A is infinite.
- Two sets A and B are *equinumerous* if there is a bijection from A to B.
- A set is *countably infinite* if it is equinumerous with $\mathcal{N}$.
- Otherwise, it is *uncountably infinite*.
- For example, $\mathcal{N}$ is countably infinite by definition, whereas $2^{\mathcal{N}}$ is uncountably infinite (to be proved later).
- A set A is countably infinite iff its elements can be enumerated.
  - (if) If $A = \{a_0, a_1, \ldots\}$, the following function $f : \mathcal{N} \to A$ is a bijection:
  $$f(i) = a_i.$$
  - (*only-if*) If there exists bijection $f : \mathcal{N} \to A$, A can be enumerated as a_0, a_1, ... where
  $$a_i = f(i)$$

- Union of two countable sets A and B are also countable:

$$A = \{a_0, a_1, \ldots\}, B = \{b_0, b_1, \ldots\}$$
$$A \cup B = \{a_0, b_0, a_1, b_1, a_2, b_2, \ldots\}$$

- Cartesian product of two countable sets A and B are also countable:

$$A \times B = \{(a_0, b_0), (a_1, b_0), (a_0, b_1), (a_1, b_1), \ldots\}$$

- Countable union of countable sets is countable.

3. Proof techniques

3.1. The Pigeonhole principle.

Lemma *If A and B are nonempty finite sets and $|A| > |B|$, then there is no one-one function from A to B.*

Proof: For any $f : A \to B$, f must not be one-one.

Basis step. Suppose $|B| = 1$ and $|A| > 1$. Then $f : A \to B$ has at least two distinct $a_1, a_2 \in A$ with $f(a_1) = f(a_2) = b$ and hence f is not one-one.

Induction hypothesis. Suppose that f is not one-one when $|A| > |B|, |B| \leq n$ for some n. Now suppose that $|A| > |B| = n + 1$.

Inductive step.

- Choose some $b \in B$. If $|f^{-1}(b)| \geq 2$, then f is not one-one.
- Otherwise, let $g : A - f^{-1}(b) \to B - \{b\}$ be defined so that $g(a) = f(a)$ for all $a \in A - f^{-1}(b)$.
- $|A - f^{-1}(b)| \geq |A| - 1$, $|B - \{b\}| = n \leq |A| - 1$. Hence, by induction hypothesis, g is not one-one.
- Since g mimics f, f is not one-one either. $\Box$

3.2. The diagonalization principle:
Let $R \subset A{\times}A$.

- Complement of the diagonal: $D = \{a : a{\in}A, (a,a){\notin}R\}$.
- Rows: $R_a = \{b : b{\in}A, (a,b){\in}R\}$.

  The complement of the diagonal is different from each row, ie for any a

  $$D{\neq}R_a.$$

Use of the diagonlization principle:

Theorem. $2^{\mathcal{N}}$ is uncountable.

Proof: Suppose $2^{\mathcal{N}}$ is countable, then there is a bijection $f : \mathcal{N} \to 2^{\mathcal{N}}$. Define $R \subset \mathcal{N}{\times}\mathcal{N}$ by

$$R = \{(i,j) : j{\in}f(i)\}.$$

Consider

$$R_i = \{j : (i,j){\in}R\} = \{j : j{\in}f(i)\} = f(i).$$

Let D be the complement of the diagonal of R. Then D is different from each $f(i)$. Hence, f is not onto. A contradiction. $\Box$