COMP 272 Theory of Computation
April 30, 1997, **Quiz Two – Solutions**

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- Answer **ALL** questions. For questions 1 and 2 you are only required to write down the answers; you do not need to justify them. For question 3 you must prove your answers.
- Time allowed: 50 minutes.
- The exam is worth 150 points. Each question is worth 50 points.

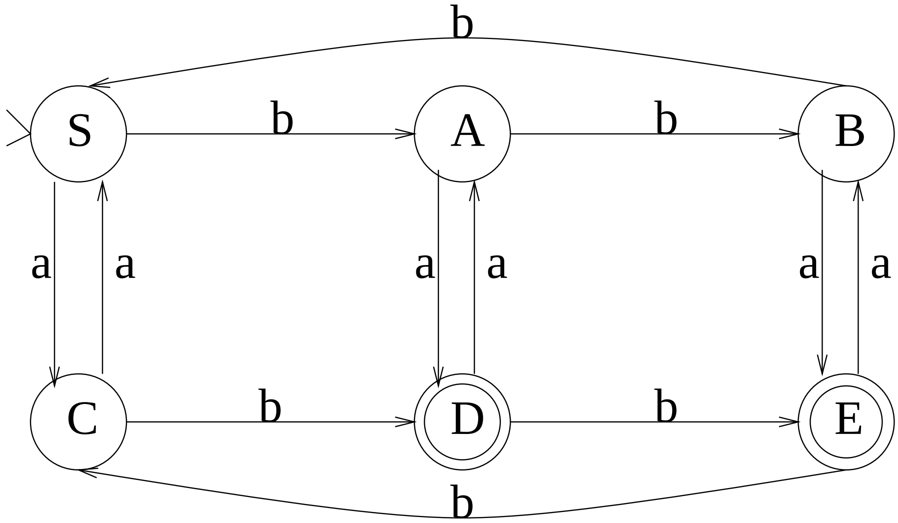
1. **Regular Grammars**

   Let

   $$L = \{w \in \{a,b\}^* \ : \ (\text{Number of } a\text{-s in } w \text{ is odd and number of } b\text{-s is not divisible by 3})\}$$

   Construct a *regular grammar* G that generates L (i.e., $L(G) = L$). Recall that a regular grammar is a special form of a context-free grammar.

   Solution: First build a DFA that accepts L and then translate it into a regular grammar that generates L by using the rules taught in class.



$$\begin{array}{ll} S \to bA & S \to aC \\ A \to bB & A \to aD \\ B \to bS & B \to aE \\ C \to bD & C \to aS \\ D \to bE & D \to aA \\ E \to bC & E \to aB \\ D \to e & E \to e \end{array}$$

2. **Context Free Languages and Pushdown Automata**

(a) Write down a Context Free Grammar that generates the following language:

$$L = \{a^i b^j c^k \ : \ i = k \text{ or } i = j\}$$

Note: the *or* is an inclusive or, i.e., all three of the following possibilities are allowed: (i) $i = k$; (ii) $i = j$, (iii) $i = j$ and $i = k$.

(b) Build a pushdown automaton that recognizes L. Recall that a pushdown automaton is of the form

$$M = (K, \Sigma, \Gamma, \Delta, s, F)$$

where Δ, the transition relation, is of the form

$$\Delta \subset (K \times \Sigma^* \times \Gamma^*) \times (K \times \Gamma^*).$$

In this case $\Sigma = \{a, b, c\}$ and you must specify all of the remaining components of M.

(a) $\Sigma = \{a, b, c\}$, $V = \{a, b, c, S, S_1, S_2, B, C\}$ and the rules are

$$\begin{aligned} S &\to S_1C|S_2, & & \\ S_1 &\to e|aS_1b, & B &\to e|bB, \\ C &\to e|cC, & S_2 &\to B|aS_2C \end{aligned}$$

(b) Set $K = \{p, q\}$, $\Gamma = \{a, b, c, S, S_1, S_2, B, C\}$ $s = p$, and $F = \{q\}$.

The transitions are

$$((p, e, e),\ (q, S))$$

$$\begin{array}{rl} ((q, e, S),\ (q, S_1C)) & ((q, e, S),\ (q, S_2)) \\ ((q, e, S_1),\ (q, e)) & ((q, e, S_1),\ (q, aS_1b)) \\ ((q, e, C),\ (q, e)) & ((q, e, C),\ (q, cC)) \\ ((q, e, B),\ (q, e)) & ((q, e, B),\ (q, bB)) \\ ((q, e, S_2),\ (q, B)) & ((q, e, S_2),\ (q, aS_2c)) \end{array}$$

$$\begin{array}{rl} ((q, a, a),\ (q, e)) & ((q, b, b),\ (q, e)) \\ ((q, c, c),\ (q, e)) & \end{array}$$

3. **Proving that languages are/are not Context Free**

(a) The language
$$L_1 = \{a^n b^n c^m \ : n \le m \le 2n\}$$
is not Context-free. Prove this fact. In your proof fully state the theorems that you are using (including their assumptions and their consequences).

(b) Let
$$L_2 = \{w \in \{a,b,c\}^* : \text{(Number of } a\text{-s in } w) = \text{(Number of } b\text{-s in } w) \text{ and (Number of } a\text{-s in } w) \le \text{(Number of } c\text{-s in } w) \le 2 \cdot \text{(Number of } a\text{-s in } w)\}$$
Is L_2 Context-free? Prove your answer. When proving your answer, fully state the theorems that you are using (including their assumptions and their consequences).

(a) We use the pumping theorem for CFLS which states that if L is a CFL then there exists K such that for all $w \in L$ with $|w| > K$: there exist u, v, x, y, z such that
(a) $w = uvxyz$
(b) $|vy| \ge 1$
(c) $\forall s \ge 0.\, uv^s xy^s z \in L$

Suppose then that L_1 is a CFL. Let K be the constant in the pumping theorem and define $w = a^K b^K c^K$. Since $|w| > K$ there exist u, v, x, y, z satisfying (a), (b) and (c).

First note that v can not contain more than one type of character because if it did, e.g., it contained a and b, then uv^2xy^2z would contain a subpattern of the form $\cdots a \cdots b \cdots a \cdots b$ which is impossible since $uv^2xy^2z \in L_1$ and strings in L_1 can not contain that subpattern. Similarly y can not contain more than one type of character.

Thus vy can contain at most two types of characters.

Set $w' = uv^{K+2}xy^{K+2}z$. We will see that regardless of the choice of v and y it is always the case that $w' \notin L_1$.

First suppose that vy contains only one type of character, i.e., only a, only b or only c. It cannot be only a because then w' would contain more a-s than b-s Similarly if it contained only b-s then w' would contain more b-s than a-s . Finally, if it contained only c-s then number of c-s in w' is greater than $2K$ where K is the number of a-s so again $w' \notin L_1$.

Thus vy contains two types of characters with v containing one type and y containing another.

Now suppose that $v = a^i$ and $y = c^j$. Then w' will contain more a-s than b-s. If $v = b^i$ and $y = c^j$ then w' contains more b-s than a-s: in both cases $w' \notin L_1$.

The only case left that we have not examined is when $v = a^i$, and $y = b^j$. But then number of a-s in w' is at least $K + i(K+1) > 2K$ where K is still the number of c-s in w' so $w' \notin L_1$.

We have thus shown that any choice of u, v, x, y, z satisfying the three conditions leads to a contradiction and thus L can not be a CFL.

(b) Suppose that L_2 is a CFL. Recall that the intersection of a CFL and a regular language is a CFL. Thus $L_2 \cap a^*b^*c^*$ is a CFL. But, $L_1 = L_2 \cap a^*b^*c^*$ and we have already seen that L_1 is not a CFL. This proves that L_2 is not a CFL.