The Knuth-Yao Quadrangle Inequality Speedup is a Consequence of Total Monotonicity

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Motivation

- Nothing new: material here goes back 20-30 years.
- There are two classic cookbook
  Dynamic Programming Speedups in the literature:
  Knuth-Yao technique & SMAWK algorithm.
- They “feel” similar. Are they related?
- Knuth-Yao predates online algorithms.
  Can the KY speedup be maintained online?
- Answers to the two questions turned out to be related.
- Note: major confusion arises in the analysis because certain essential terms, e.g., quadrangle-inequality, monotone and online-algorithm have been used in very different ways in the two techniques’ literature.

Outline

- Background
  - Knuth-Yao (KY) Quadrangle Inequality (QI) Speedup
  - SMAWK Algorithm for finding
    Row Minima of Totally Monotone (TM) Matrices
- The D^d Decomposition
  A transformation from QI to TM such that
  SMAWK solves KY problem as quickly as KY.
- The L^m and R^m Decompositions
  Another transformation from QI to TM that
  (1) implies KY speedup and (2) enables online solution.
- Extensions
  Applying the technique to known generalizations of KY.

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- Both techniques are often used to speed up DPs.
- How are the two techniques related?

Quadrangle Inequality

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    p_l, $(l = 1 \ldots n)$ is the weight that search-key $=$ Key$_l$

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  - $n+1$ external nodes corresponds to unsuccessful search
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    q_l, $(l = 0 \ldots n)$ is the weight that Key$_l$ $<$ search-key $<$ Key$_{l+1}$
  - Minimize the number of comparisons

$$\sum_{1 \le l \le n} p_l \cdot (1 + \underbrace{d(p_l)}_{\text{depth}}) + \sum_{0 \le l \le n} q_l \cdot \underbrace{d(q_l)}_{\text{depth}}$$

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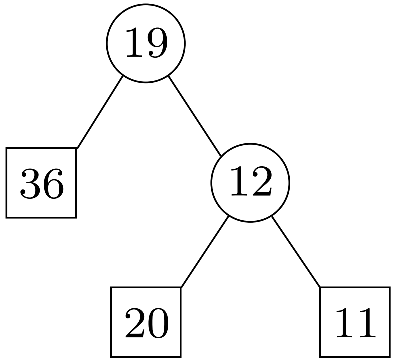

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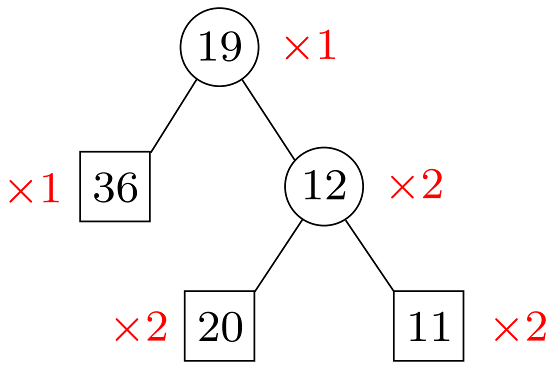

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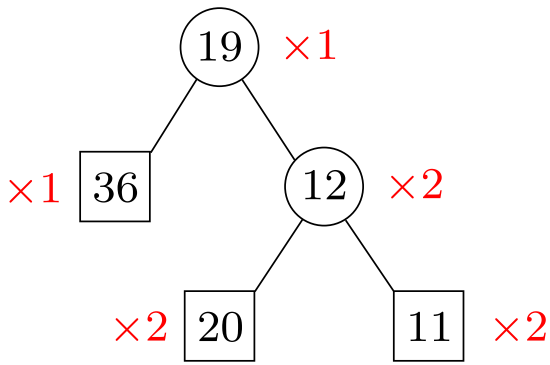

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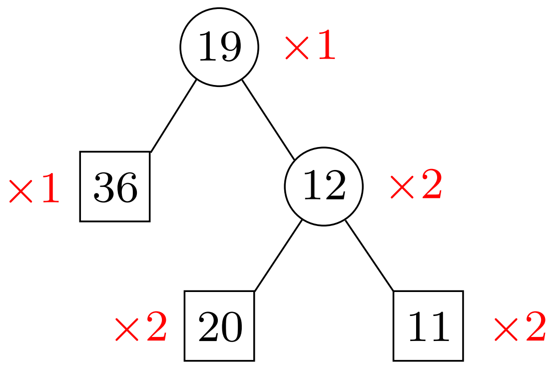

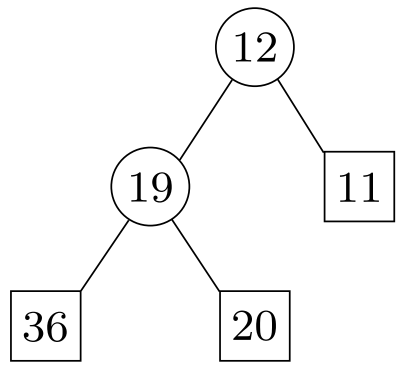

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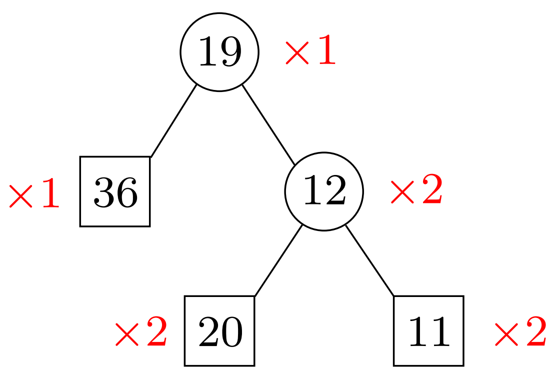
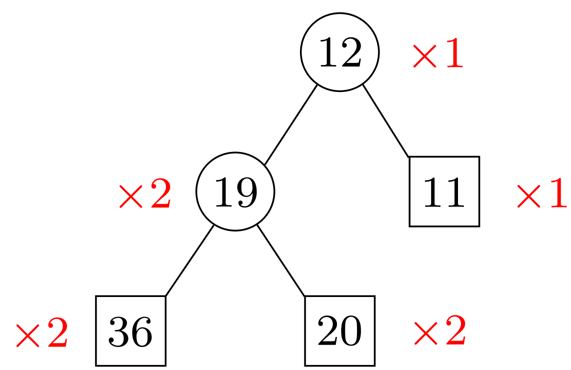


Cost $= 141$



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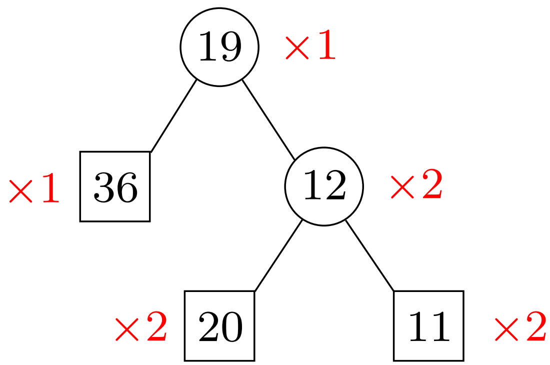

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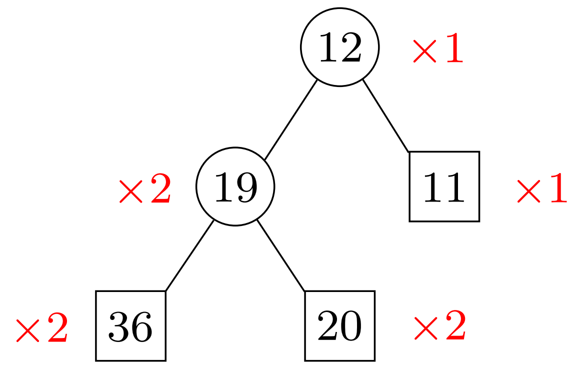

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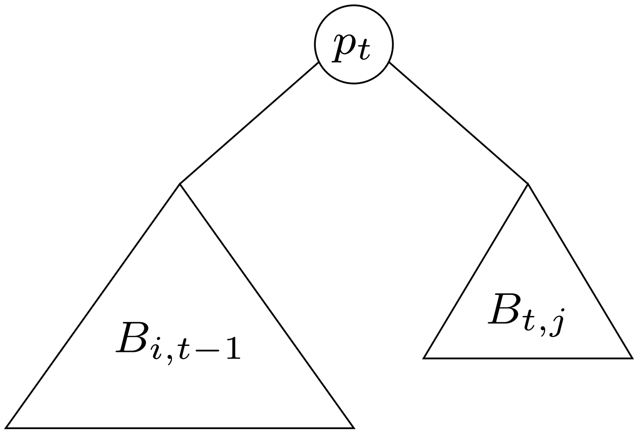
  - $B_{i,j}$ the optimal BST for the subproblem $\text{Key}_{i+1}, \ldots, \text{Key}_j$
  - DP recurrence

$$B_{i,j} = \sum_{l=i+1}^{j} p_l + \sum_{l=i}^{j} q_l + \min_{i<t\leq j}\{B_{i,t-1} + B_{t,j}\}$$



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- DP: Straightforward Calculation

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	0	1	2	3	4	5	6
0	0						
1		0					
2			0				
3				0			
4					0		
5						0	
6							0

$B_{i,j}$ depends on the entries to the left and below.

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	0	1	2	3	4	5	6
0	0	230					
1		0	146				
2			0	75			
3				0	43		
4					0	44	
5						0	52
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	0	1	2	3	4	5	6
0	0	230	433				
1		0	146	260			
2			0	75	141		
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	0	1	2	3	4	5	6
0	0	230	433	586			
1		0	146	260	349		
2			0	75	141	250	
3				0	43	119	204
4					0	44	121
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	0	1	2	3	4	5	6
0	0	230	433	586	698		
1		0	146	260	349	491	
2			0	75	141	250	357
3				0	43	119	204
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	0	1	2	3	4	5	6
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- Naive: $O(n^3) = \sum_{i=1}^{n} \sum_{j=i}^{n} \Theta(j-i)$

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$$K_B(i,j) \leq K_B(i,j+1) \leq K_B(i+1,j+1)$$

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	i	$i+1$
j	$K_B(i,j)$	$K_B(i,j+1)$
$j+1$		$K_B(i+1,j+1)$

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- The index table

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	0	1	2	3	4	5	6
0		0					
1			1				
2				2			
3					3		
4						4	
5							5
6							

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- The index table

	0	1	2	3	4	5	6
0		0	0				
1			1	1			
2				2	2		
3					3	4	
4						4	5
5							5
6							

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  - $O(n^2)$ total work over all n diagonals.

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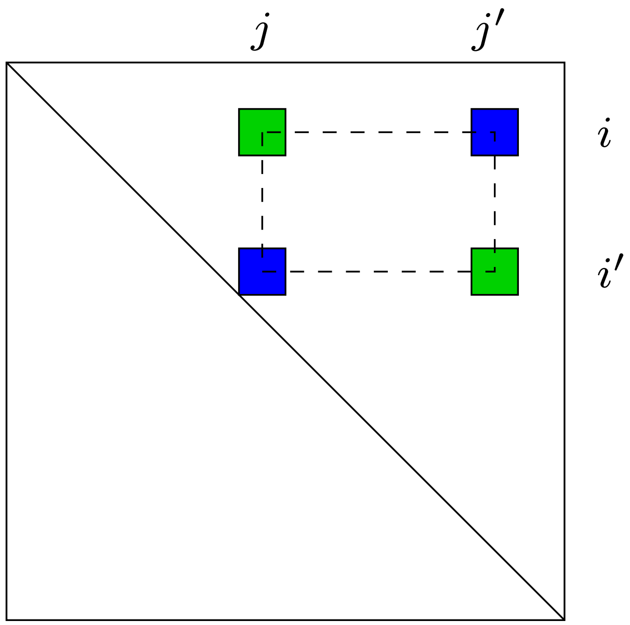

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    $$f(i,j) + f(i',j') \leq f(i',j) + f(i,j')$$
  - Function $f(i,j), (0 \leq i \leq j \leq n)$ is Monotone over the integer lattice (MIL), if $\forall [i,j] \subseteq [i',j']$
    $$f(i,j) \leq f(i',j')$$

Speedup using Quadrangle Inequality

$$B_{i,j} = w(i,j) + \min_{i<t\leq j}\{B_{i,t-1} + B_{t,j}\}$$

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  - $\Rightarrow$ exactly Knuth's result.

Online Problem

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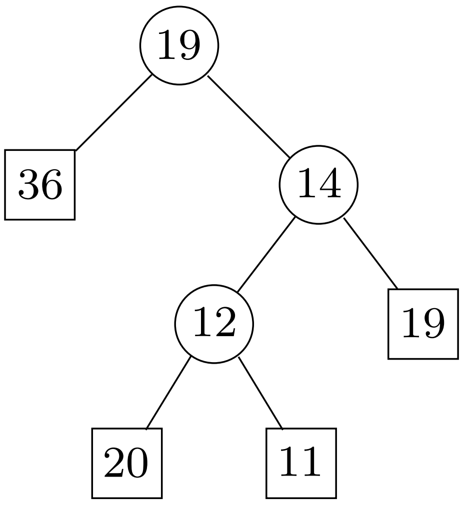
$p = (\quad 19, 12, 14 \quad) \quad q = (\quad 36, 20, 11, 19 \quad)$

	1	2	3	4	5	6
1						
2		0	75	141	250	
3			0	43	119	
4				0	44	
5					0	
6						



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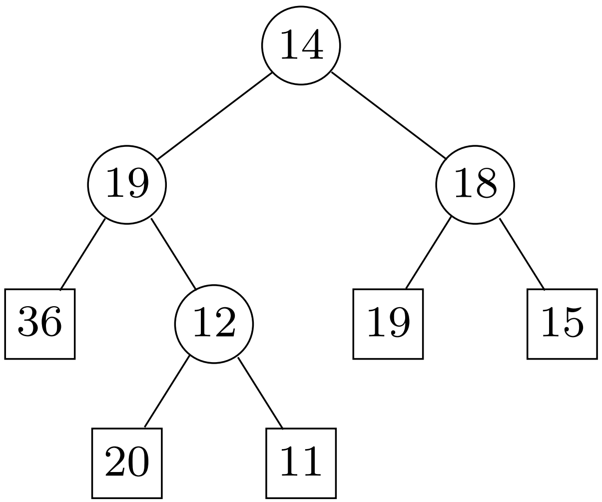
	1	2	3	4	5	6
1						
2		0	75	141	250	357
3			0	43	119	204
4				0	44	121
5					0	52
6						0



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$p = (\quad 21, 19, 12, 14, 18) \quad q = (\quad 89, 36, 20, 11, 19, 15)$

	1	2	3	4	5	6
1	0	146	260	349	491	624
2		0	75	141	250	357
3			0	43	119	204
4				0	44	121
5					0	52
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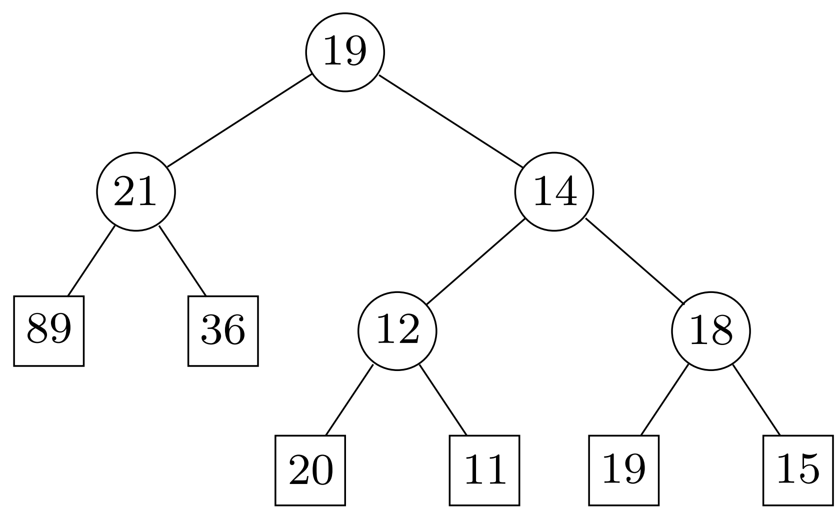

Outline

- Background
  - Knuth-Yao (KY) Quadrangle Inequality (QI) Speedup
  - SMAWK Algorithm for finding
    Row Minima of Totally Monotone (TM) Matrices
- The D^d Decomposition
  A transformation from QI to TM such that
  SMAWK solves KY problem as quickly as KY.
- The L^m and R^m Decompositions
  Another transformation from QI to TM that
  (1) implies KY speedup and (2) enables online solution.
- Extensions
  Applying the technique to known generalizations of KY.

Totally Monotone Matrices

- Definition

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7	2	4	3	8	9
5	1	5	1	6	5
7	1	2	0	3	1
9	4	5	1	3	2
8	4	5	3	4	3
9	6	7	5	6	5

$\text{RM}_M(1) = 2$

$\text{RM}_M(2) = 4$

$\text{RM}_M(3) = 4$

$\text{RM}_M(4) = 4$

$\text{RM}_M(5) = 6$

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Totally Monotone Matrices

- Definition (Cond.)
  - A 2×2 Monotone matrix

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4	5

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- SMAWK was culmination of decade(s) of work on similar problems; speedups using convexity and concavity.
- Has been used to speed up many DP problems, e.g., computational geometry, bioinformatics, k-center on a line, etc.

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  TM property is often established via Monge property.

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The Monge Property

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Function $f(i, j)$

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    - QI: $f(i,j)$ is function to be calculated.
      Need all $f(i,j)$ entries.
    - Monge: $M_{i,j}$ implicitly given.
      Only need the row minima, but not other entries.

Monge Property

$$\forall i \le i' \quad \forall j \le j' \qquad M_{i,j} + M_{i',j'} \le M_{i',j} + M_{i,j'}$$

- Theorems

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    - $\Rightarrow$ Only need to prove Monge property for adjacent rows and columns.

Monge Property

- General Scheme

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  1. Prove Monge Property for adjacent rows and columns

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  1. Prove Monge Property for adjacent rows and columns
  2. (Automatically implies) Monge Property
  3. (Automatically implies) Totally Monotone Property
  4. Use SMAWK algorithm to find row minima
  5. Usually $\Theta(n)$ speedup

Relationship?

Quadrangle Inequality Totally Monotone (Monge)

Relationship?

Quadrangle Inequality	Totally Monotone (Monge)
A matrix to be calculated	A matrix given implicitly

Relationship?

Quadrangle Inequality

A matrix to be calculated

Need all $O(n^2)$ entries

Totally Monotone (Monge)

A matrix given implicitly

Need only $O(n)$ row minima

Relationship?

Quadrangle Inequality

A matrix to be calculated

Need all $O(n^2)$ entries

$O(n^3)$ to $O(n^2)$ speedup

Totally Monotone (Monge)

A matrix given implicitly

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$O(n^2)$ to $O(n)$ speedup

Relationship?

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Totally Monotone (Monge)

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- This talk

Relationship?

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  - QI instance is decomposed into $\Theta(n)$ TM instances
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  - $\Rightarrow$ QI instance requires $O(n^2)$ time in total

Decompositions

QI instance $\longrightarrow$ $\Theta(n)$ TM instances

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- L^m and R^m decompositions

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  - Permits using algorithm of [Larmore, Schieber (1990)], to get "online" KY speedup.

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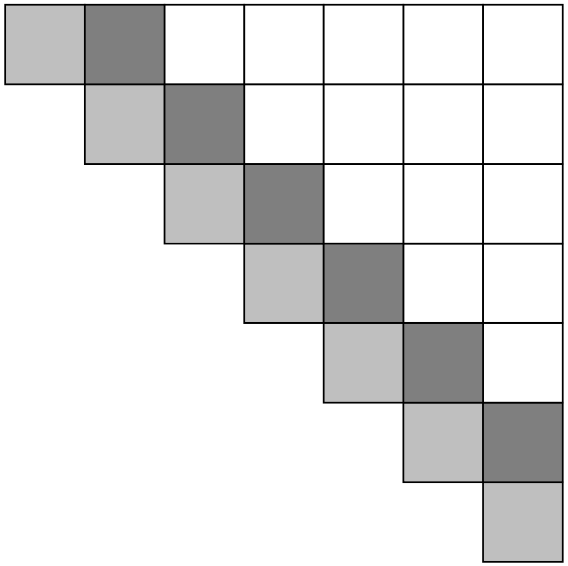

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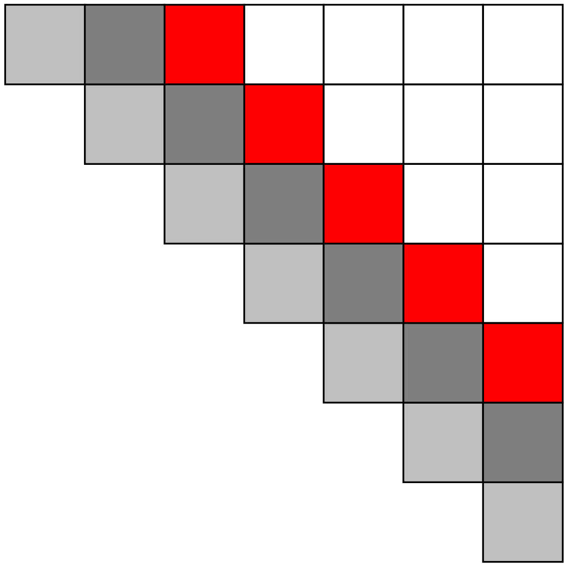

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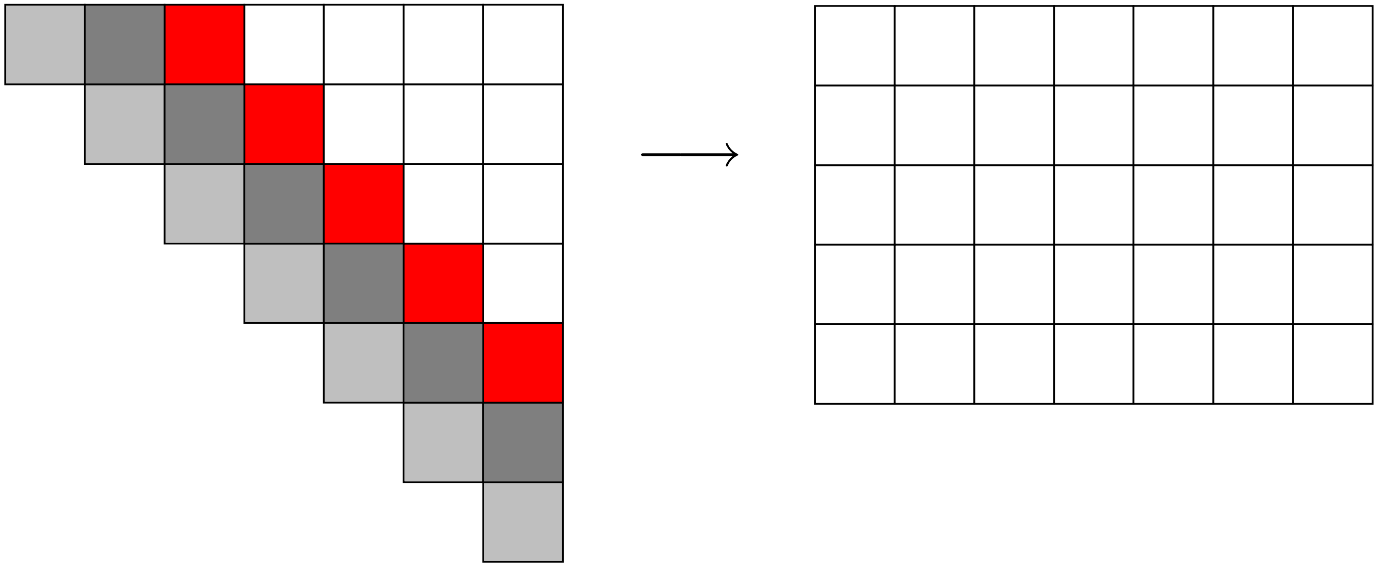

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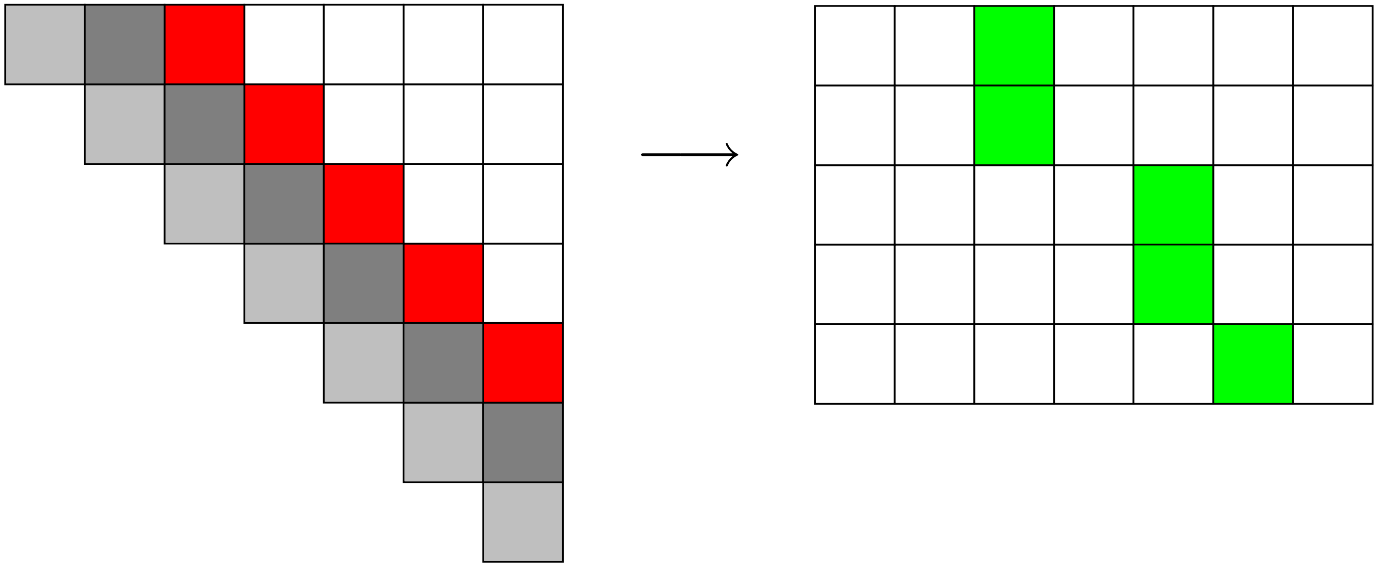

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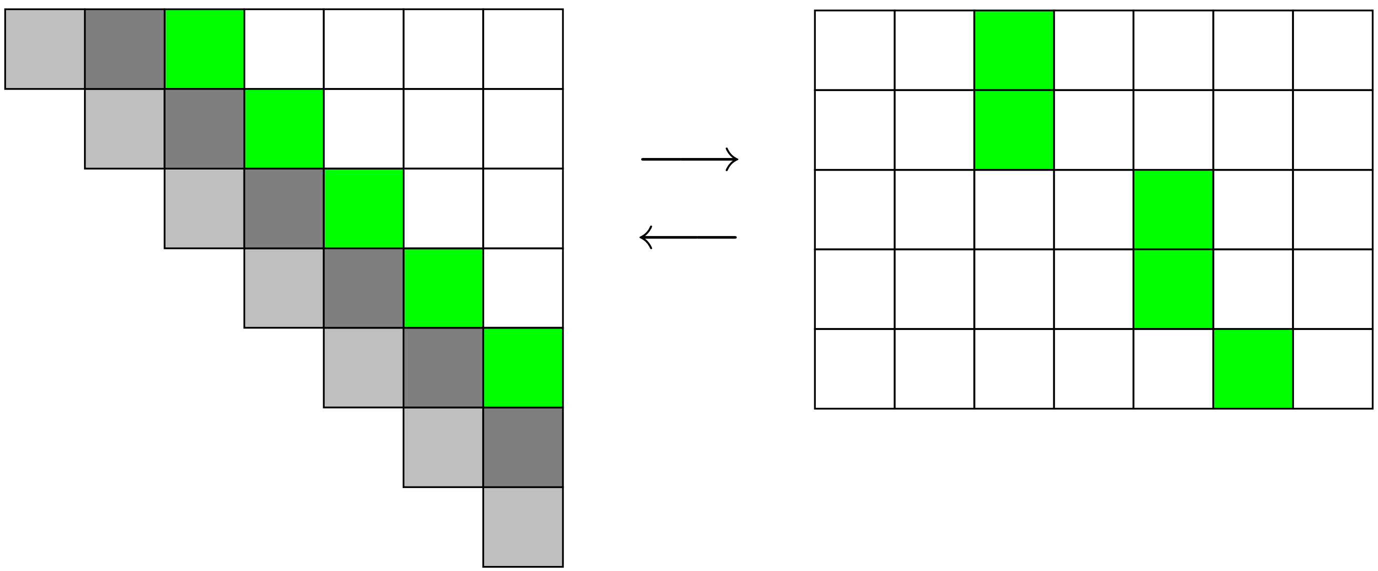

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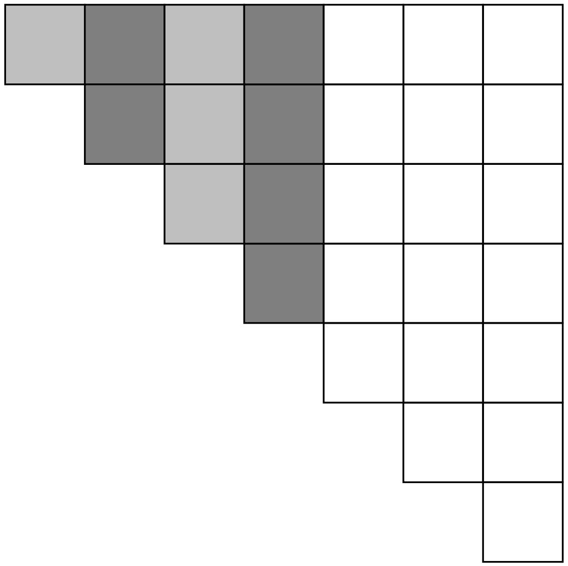

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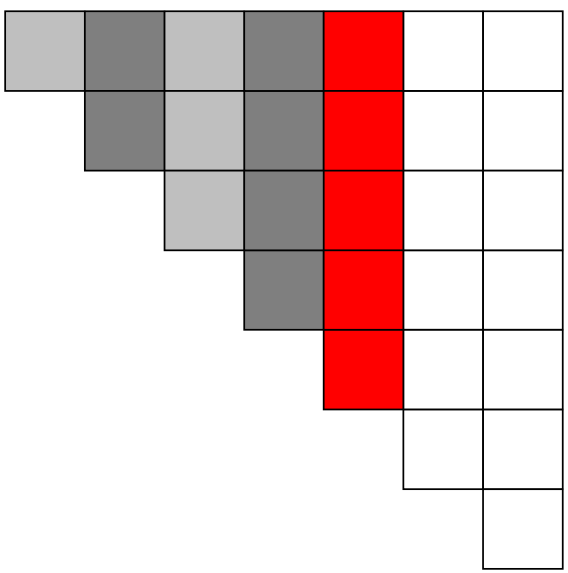

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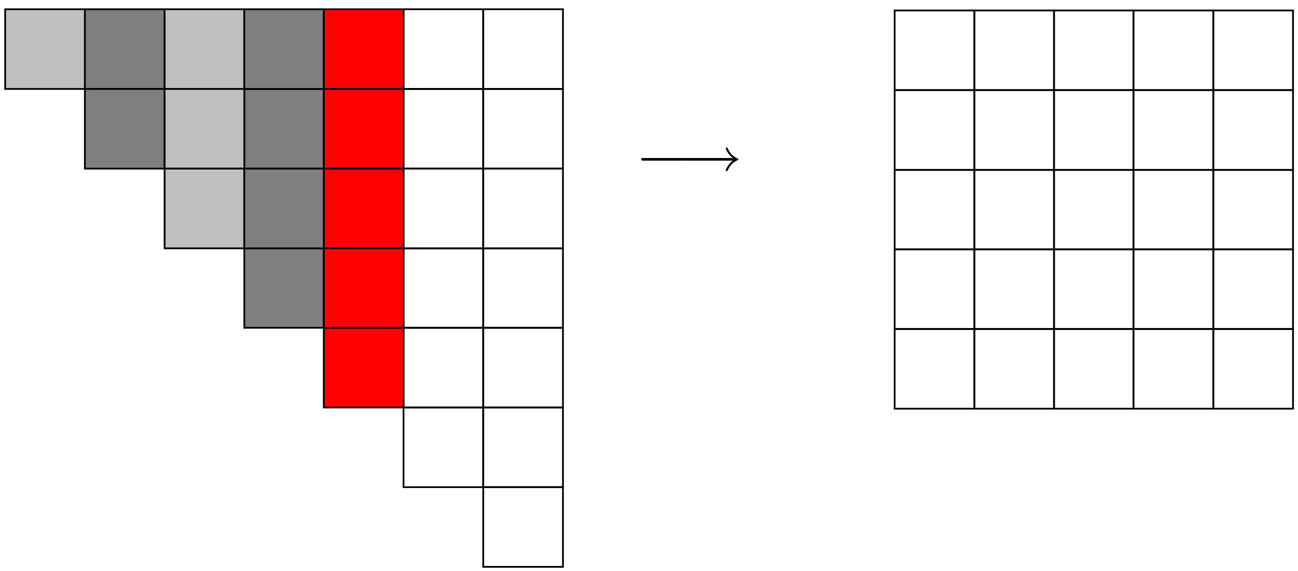

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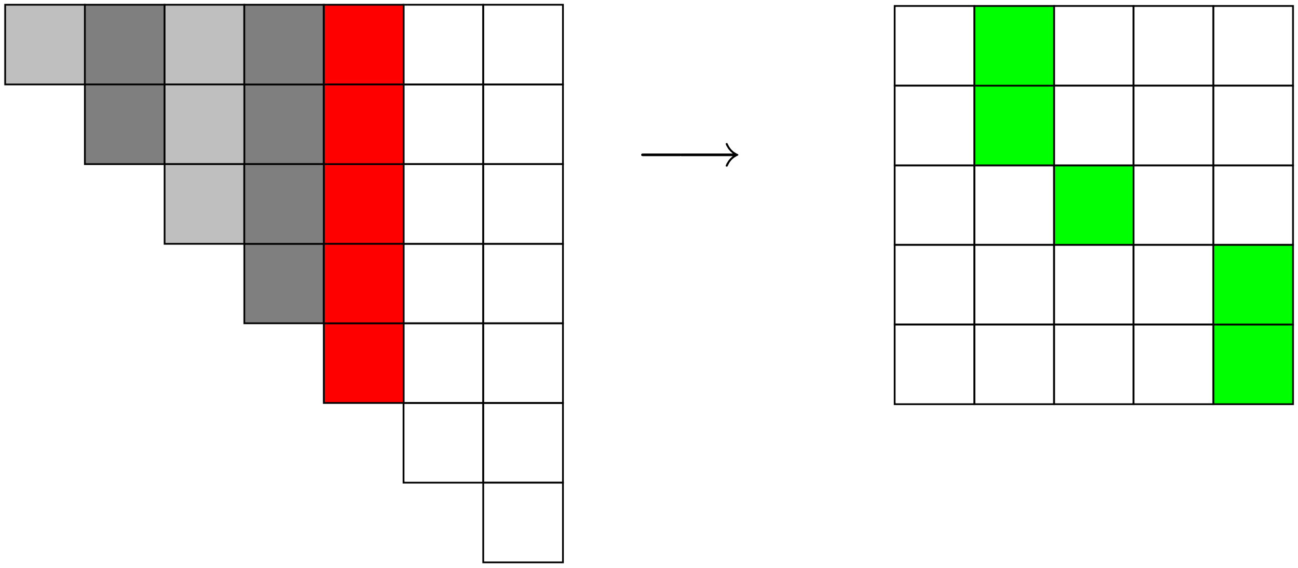

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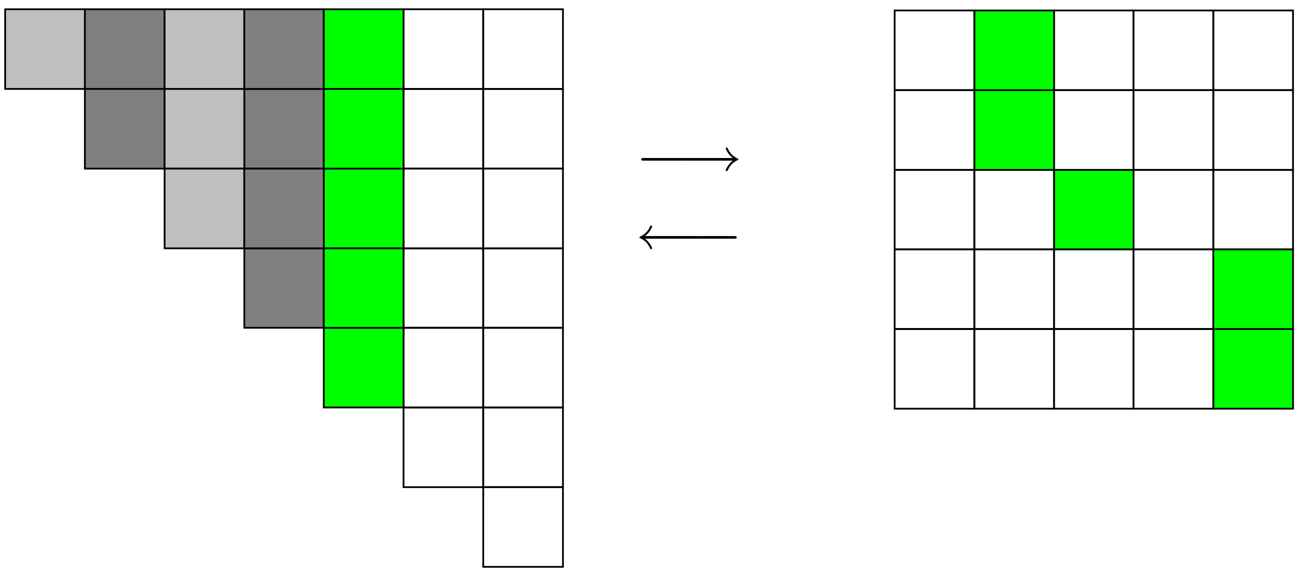

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  - Knuth-Yao (KY) Quadrangle Inequality (QI) Speedup
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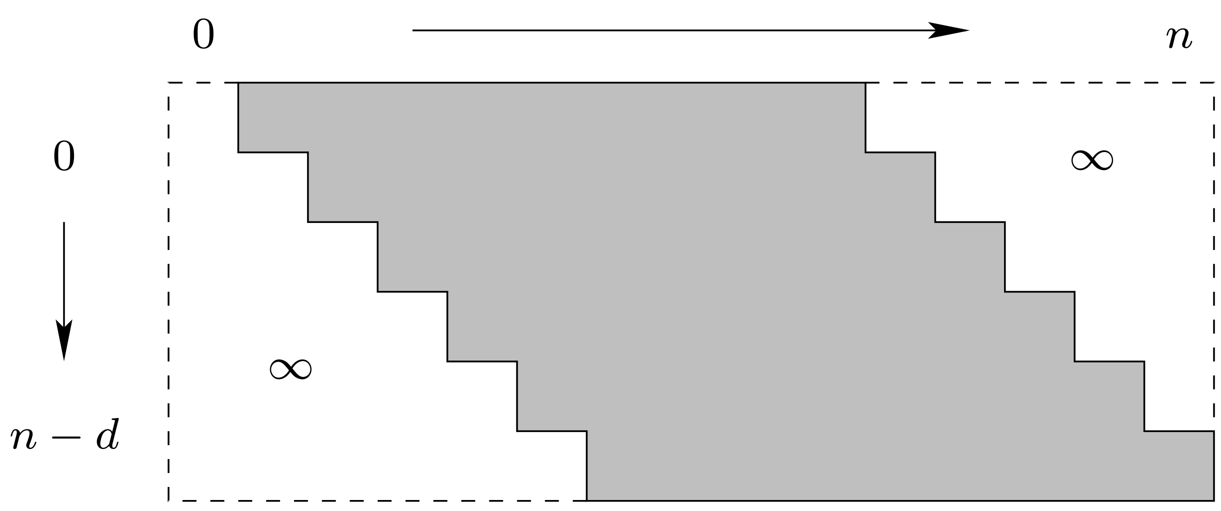

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- Note: Must run SMAWK on D^d in the order $d = 1, 2, 3, \ldots$

  Entries in D^d depend upon row minima of $D^{d'}$ where $d' < d$.

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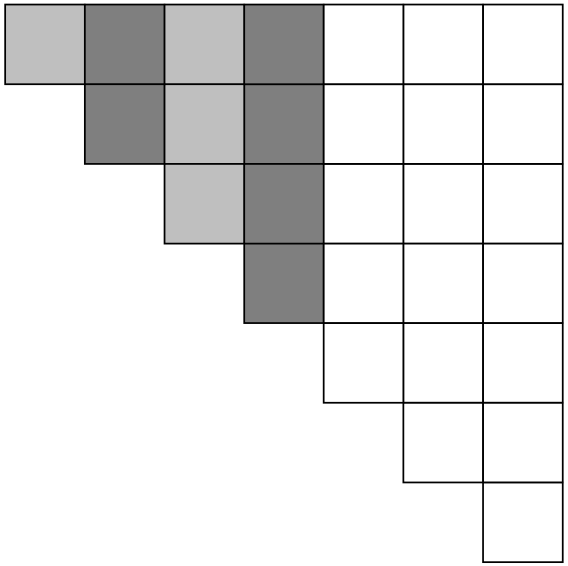

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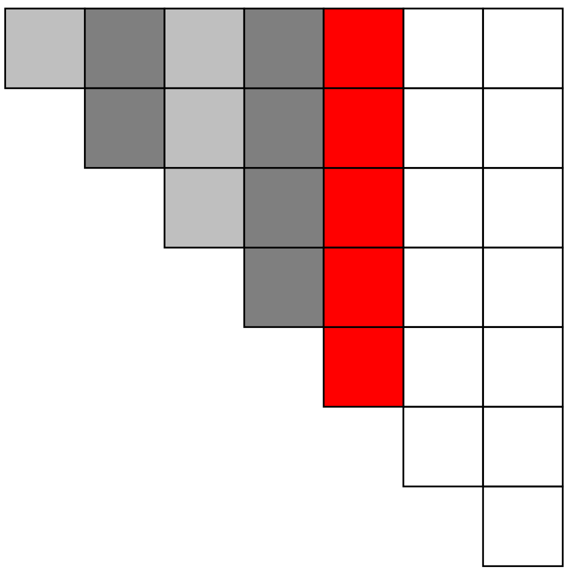

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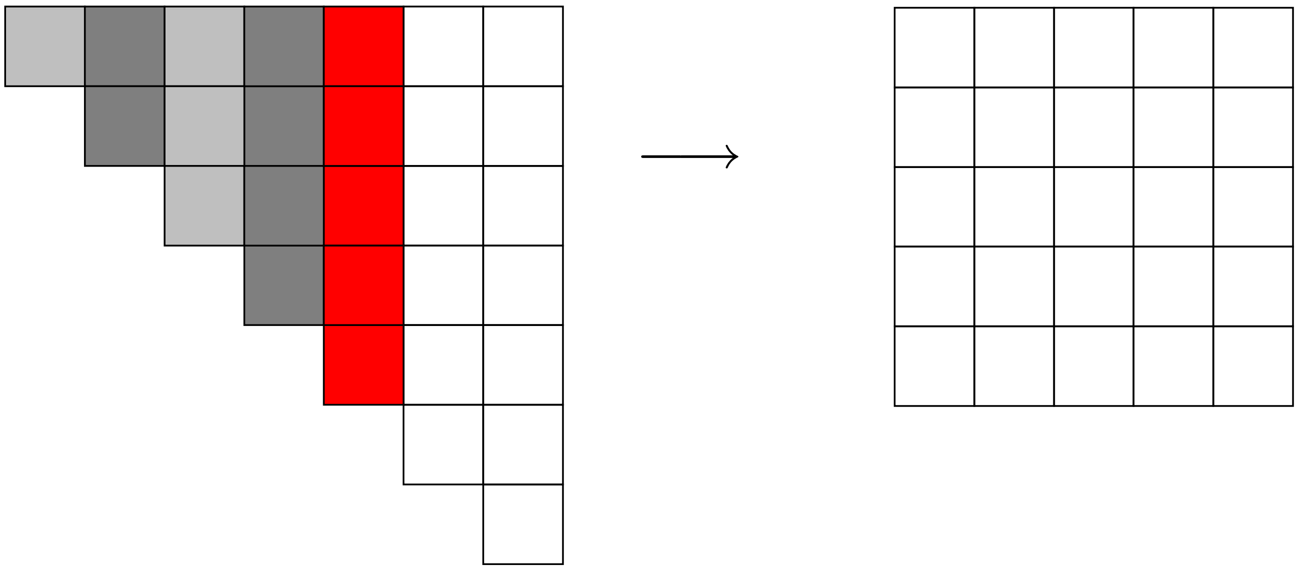

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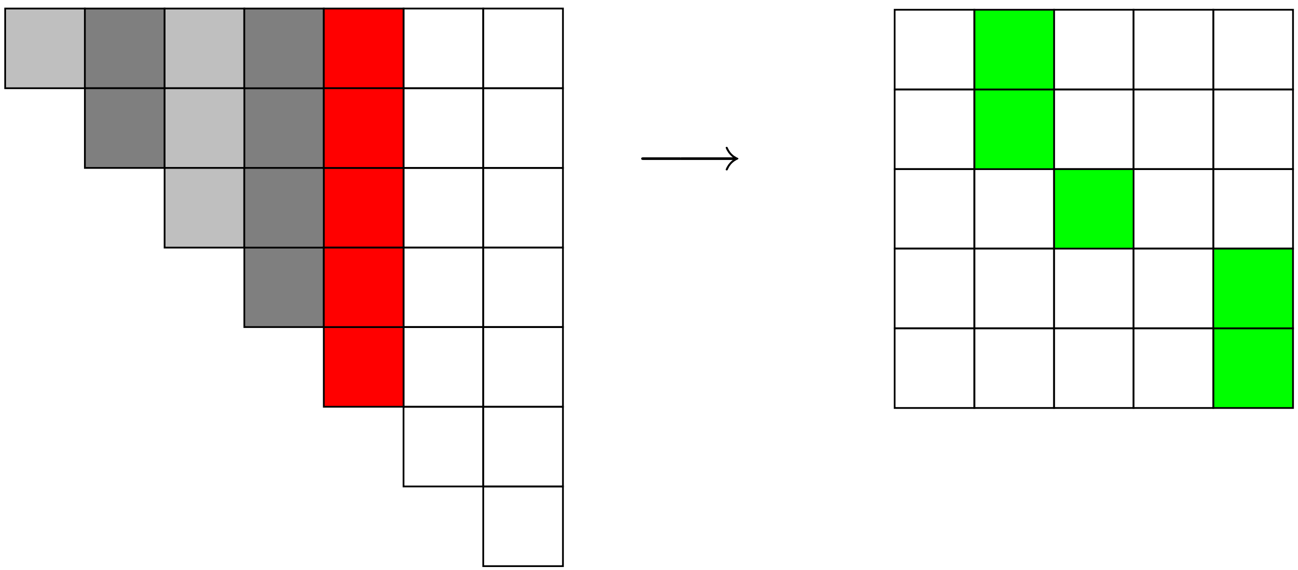

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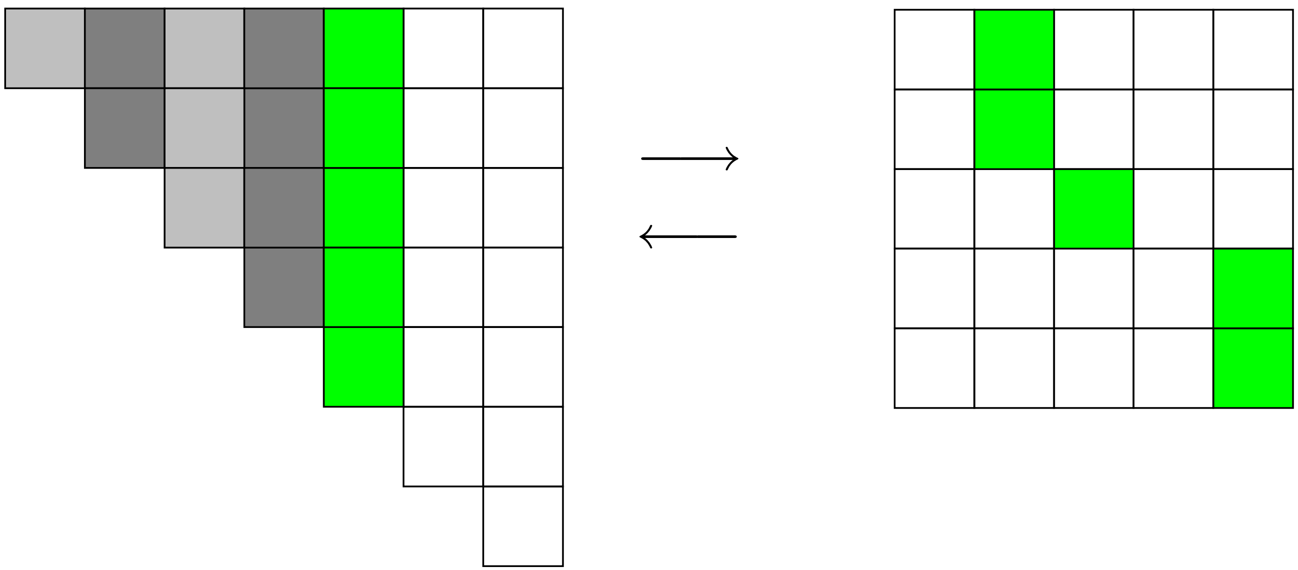

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    $$B_{i,m} = \min_{i<j\le m} R^m_{i,j} = \min_{0<j\le m} R^m_{i,j}$$

- Lemma
  - R^m is Monge, for each $1 \le m \le n$.

R^m Decomposition

$$R^m_{i,j} = \begin{cases} w(i,m) + \{B_{i,j-1} + B_{j,m}\} & \text{if } 0 \leq i < j \leq m \\ \infty & \text{otherwise} \end{cases}$$

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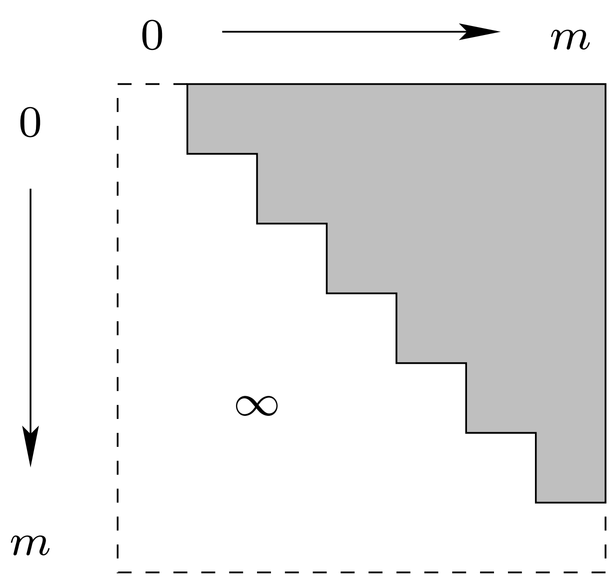

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So

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L^m and R^m Imply Original KY Result

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1	1	2	2	2	2	RM$_M(1) = 2$
1	1	1	1	2	2	RM$_M(2) = 4$
1	1	1	1	2	2	RM$_M(3) = 4$
1	1	1	1	2	2	RM$_M(4) = 4$
1	1	1	1	1	1	RM$_M(5) = 6$
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- $L^m \longrightarrow K_B(i,j) \le K_B(i,j+1)$
  - Similar

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  - $D^d_{i,j} = w(i, i+d) + \{B_{i,j-1} + B_{j,i+d}\} \qquad (0 \leq i < j \leq i+d \leq n)$

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  - $O(n)$ time for each column $\Rightarrow O(n^2)$ in total.

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Finding row minima in totally monotone matrices with limited dependency.
This is also known as online TM problem

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Entries of column j can depend on the row minima of rows i where $M_{i,j} = \infty$.

Green: the column j.
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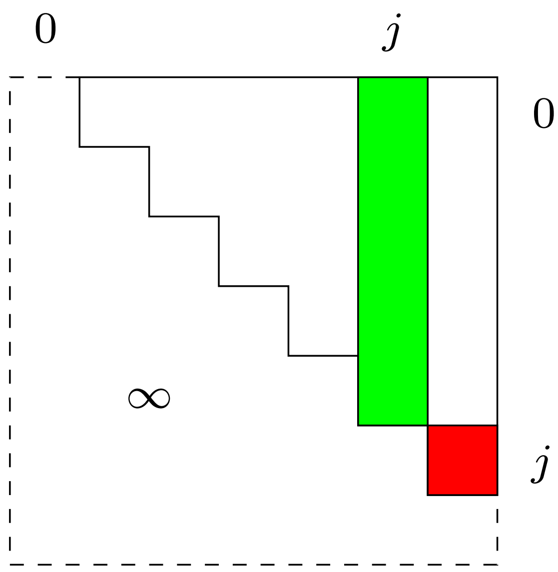

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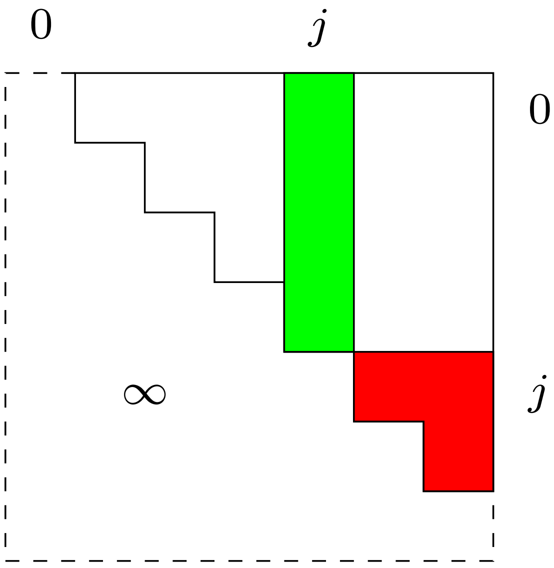

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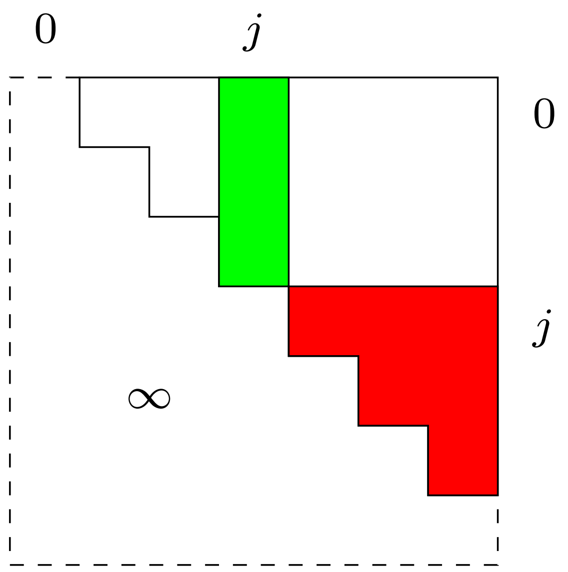

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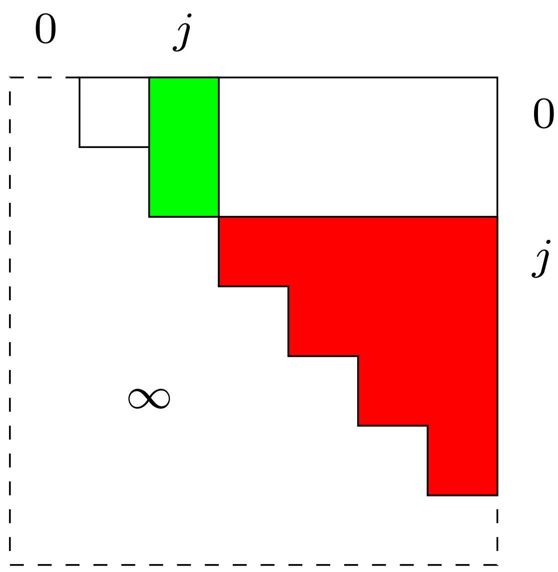

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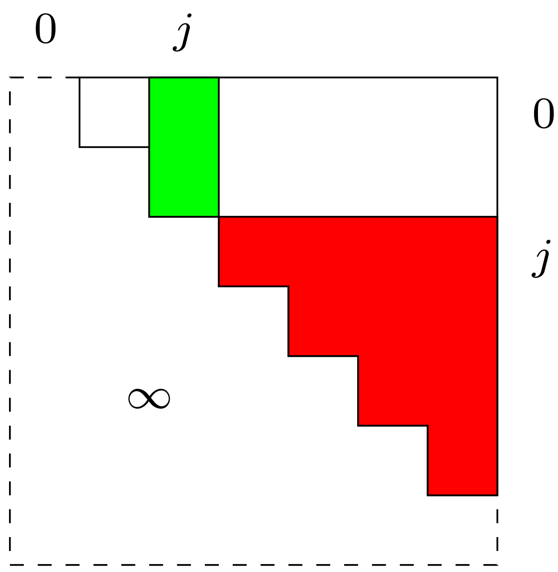

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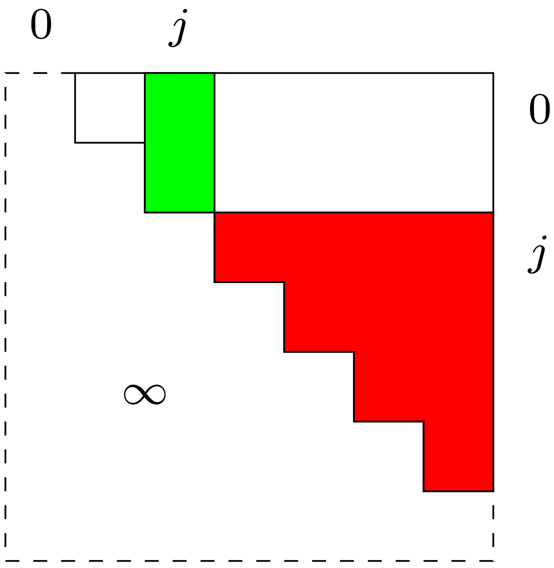

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R^m satisfies the condition of LARSCH.

Note

Aggarwall and Park (FOCS '88) developed a 3-D monotone matrix representation of the $K - Y$ problem and then showed how to use an algorithm due to Wilber (for *online computation of maxima of certain concave sequences*) to calculate "tube-maxima" of their matrices.

Careful decomposition of their work yields a decomposition similar to L^m and an $O(n)$ algorithm for calculating its row-minima. This provides an alternative derivation of the previous result (with a symmetry argument extending it to R^m)

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  - Current step: Optimal BST for $\text{Key}_{l+1}, \ldots, \text{Key}_r$

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	1	2	3	4	5	6
1	0	146	260	349	491	624
2		0	75	141	250	357
3			0	43	119	204
4				0	44	121
5					0	52
6						0

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and $\forall i < t \leq t' \leq j \leq j'$ and $i \leq i' < t'$

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- If the value of $w(i,t,j)$ is independent of t, the Borchers and Gupta definition becomes the original Knuth-Yao definition.

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    if $\forall [i,j] \subseteq [i',j']$ and $i < t \leq j$, $w(i,t,j) \leq w(i',t,j')$.

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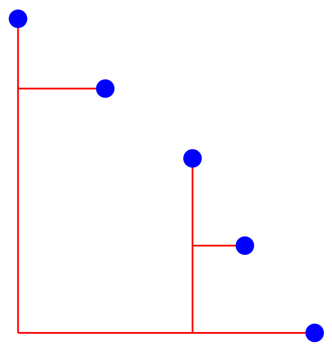

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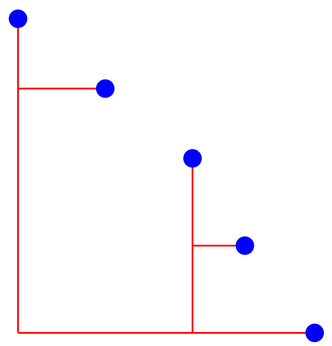

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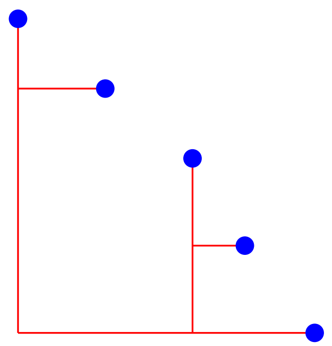

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  - $w(i,t,j)$ satisfies generalized QI and MIL.

Outline

- Background
  - Knuth-Yao (KY) Quadrangle Inequality (QI) Speedup
  - SMAWK Algorithm for finding
    Row Minima of Totally Monotone (TM) Matrices
- The D^d Decomposition
  A transformation from QI to TM such that
  SMAWK solves KY problem as quickly as KY.
- The L^m and R^m Decompositions
  Another transformation from QI to TM that
  (1) implies KY speedup and (2) enables online solution.
- Extensions
  Applying the technique to known generalizations of KY.

Questions?